Discrete Applied Mathematics 158 (2010) 1158–1180

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Covering and radius-covering arrays: Constructions and classification

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ARTICLE INFO

Article history:
Received 12 October 2009
Received in revised form 3 March 2010
Accepted 16 March 2010
Available online 13 April 2010

Keywords:
Covering array
Surjective code
Simulated annealing
Symbol fusion
Classification of codes
Partition matrix

ABSTRACT

The minimum number of rows in covering arrays (equivalently, surjective codes) and radius-covering arrays (equivalently, surjective codes with a radius) has been determined precisely only in special cases. In this paper, explicit constructions for numerous best known covering arrays (upper bounds) are found by a combination of combinatorial and computational methods. For radius-covering arrays, explicit constructions from covering codes are developed. Lower bounds are improved upon using connections to orthogonal arrays, partition matrices, and covering codes, and in specific cases by computation. Consequently for some parameter sets the minimum size of a covering array is determined precisely. For some of these, a complete classification of all inequivalent covering arrays is determined, again using computational techniques. Existence tables for up to 10 columns, up to 8 symbols, and all possible strengths are presented to report the best current lower and upper bounds, and classifications of inequivalent arrays.

1. Introduction

In the present paper, we formulate the notion of covering array in a more general manner than has been done previously. An $M \times s$ array *r-covers* an s-tuple if at least one row of the array differs from the tuple in at most r coordinate places. A *radius-covering array* $\mathrm{CA}_r(M; s, n, q)$ is an $M \times n$ array such that every $M \times s$ subarray r-covers all s-tuples from q symbols. When $r = 0$, we omit the subscript, and recover the standard, more restricted definition: A *covering array* $\mathrm{CA}(M; s, n, q)$ is an $M \times n$ array such that every $M \times s$ subarray contains as a row each s-tuple from q symbols at least once. The parameter s is the *strength* of the array.

Treating rows as tests or experimental runs, covering arrays can be applied to software and hardware testing problems and to similar problems in which interactions among columns are to be covered in as few tests as possible. The smallest possible number of tests is determined as the minimum of M for a $\mathrm{CA}(M; s, n, q)$. When this minimum is too big to permit the tests to be completed within a reasonable amount of time, it may be useful to consider the arrays $\mathrm{CA}_r(M; s, n, q)$ for $r \geq 1$, and find the fewest tests for these arrays. In this way, a considerable reduction in the smallest number of tests can be realized (at the expense of the thoroughness of the testing process, of course).

We denote by $\mathrm{CAN}(s, n, q)$ the minimum M for which $\mathrm{CA}(M; s, n, q)$ exists, and similarly by $\mathrm{CAN}_r(s, n, q)$ the minimum M for which $\mathrm{CA}_r(M; s, n, q)$ exists. When $r = 0$ every s-tuple is covered the same number λ of times, a covering array is an *orthogonal array* (OA); see [22]. The value λ is the *index* of the OA; when an OA exists with $\lambda = 1$, it necessarily has $\mathrm{CAN}(s, n, q)$ rows.

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2010.03.008

If two or more rows in a covering array completely agree, then M can easily be reduced by omitting the duplicate rows. So it is enough to study covering arrays that do not have identical rows. With this restriction, the notion of covering array is equivalent to the notion of surjective code. Some arguments can be presented more easily with surjective codes than with covering arrays. We give a brief summary of the coding theoretic background for surjective codes next.

Let Z denote a finite set of arbitrary symbols. A nonempty subset C of Z^n is a *code* of *length n* over the alphabet Z. Vectors of Z^n are *words* and vectors belonging to a code are *codewords*. A *binary* code is a code over an alphabet of 2 symbols, say $Z = \{0, 1\}$. An *s-surjective code* is a code over an alphabet of q symbols with the property that, in every s coordinate positions, all q^s possibilities occur at least once. An *s-surjective code with radius r* is a code over an alphabet of q symbols with the property that, in every set of s coordinates $i_1, i_2, \ldots, i_s$ of C and every s-tuple $(x_1, x_2, \ldots, x_s) \in Z_q^s$, there is a codeword $c \in C$ such that $c_{i_j} = x_j$ for at least $s - r$ coordinates from $1 \le j \le s$. When $s = n$, an s-surjective code with radius r is also a *covering code* and r is its *covering radius*. Covering codes are employed in Section 7.

For an introduction to coding theory, see [35]; for covering codes, see [10]. While surjective codes have been studied more extensively, the more general notion of *surjective code with radius r* was introduced in [26]. For two (somewhat out-of-date) surveys on covering arrays, see [12,21].

2. Covering arrays from Steiner systems

A *Steiner system* $S(t, k, v)$ is a pair $(V, \mathcal{B})$ where V is a set of v points, and $\mathcal{B}$ is a set of k-element subsets of V (called *blocks*) with the property that every t-element subset of V occurs as a subset of exactly one block.

Lemma 2.1. $\mathrm{CAN}(7, 12, 2) \le 264$*;* $\mathrm{CAN}(7, 11, 2) \le 242$*;* $\mathrm{CAN}(7, 10, 2) \le 222$*;* $\mathrm{CAN}(6, 9, 2) \le 111$*;* $\mathrm{CAN}(5, 10, 2) \le 56$*;* $\mathrm{CAN}(5, 9, 2) \le 54$*; and* $\mathrm{CAN}(5, 8, 2) \le 52$*. For strengths* 5 *and* 7*, each has two disjoint rows.*

Proof. There exists a Steiner system $S(5, 6, 12)$, which has 132 blocks. The characteristic vectors of its blocks form a 132×12 array A_1. Form a 66×12 matrix A_2 from the $\binom{12}{2} = 66$ characteristic vectors of sets of size two, and let A_3 be the complement of A_2. Form a 264×12 matrix A by vertically juxtaposing A_1, A_2, and A_3. We claim that A is a $\mathrm{CA}(264; 7, 12, 2)$. To verify this, consider the 7-tuple that contains a 1 entry in the columns of A and a 0 entry in the columns of A'. Each column corresponds to a point of the Steiner system. Now if $|A| \le 2$, the tuple is covered in a row of A_2; and if $|A| \ge 5$ then $|A'| \le 2$ and the tuple is covered in A_3. So we need only treat cases when $3 \le |A| \le 4$.

When A has four points and A' has three, there are four blocks B_1, B_2, B_3, B_4 for which $A \subset B_i$ for $1 \le i \le 4$. Because the Steiner system has $t = 5$, $B_i \cap B_j = A$ for $1 \le i < j \le 4$, and hence A' intersects at most three of the $\{B_i\}$ nontrivially. The block B_j for which $A' \cap B_j = \emptyset$ produces a row in which all columns in A contain 1 and all columns in A' contain 0. When A has three points and A' has four, select the 12 blocks that contain A, and delete the elements of A from each to form an $S(2, 3, 9)$. The points of A' cannot meet every block of the $S(2, 3, 9)$; let A'' be the remaining five points, and count triples involving a point of A' and a point of A''. Because there are 20 pairs with one element in A' and the other in A'', there are 10 such triples. Then because there is at most one triple induced on A', there is at least one induced on A''. The block of the $S(5, 6, 12)$ from which it arose yields a row with the three columns indexed by A containing 1, and the four indexed by A' containing 0. Hence all 7-tuples are covered and $\mathrm{CAN}(7, 12, 2) \le 264$.

Then $\mathrm{CAN}(7, 11, 2) \le 242$ and $\mathrm{CAN}(7, 10, 2) \le 222$ are obtained by removing one or two columns from A_1 retaining all rows, and employing $\binom{11}{2}$ or $\binom{10}{2}$ rows for each of A_2 and A_3.

To establish that $\mathrm{CAN}(6, 9, 2) \le 111$, form C_1 by selecting the 66 rows of A_1 that contain a 1 in the last column, and then delete the last column. Delete i further columns from C_1, and adjoin all $\binom{11-i}{2}$ rows with exactly $9 - i$ 1 entries, and all $11 - i$ rows with exactly one 1 entry. The result is a $\mathrm{CA}\left(66 + \binom{11-i}{2} + 11 - i; 6, 11 - i, 2\right)$; for $i = 2$, it establishes that $\mathrm{CAN}(6, 9, 2) \le 111$.

Now form D_1 by selecting the 36 rows of C_1 that contain a 0 in the last column, and deleting this column. Then D_1 is a 36×10 matrix. Delete i further columns from D_1, and adjoin all $10 - i$ rows with exactly $9 - i$ 1 entries, and all $10 - i$ rows with exactly one 1 entry. The result is a $\mathrm{CA}(36 + 2(10 - i); 5, 10 - i, 2)$; hence $\mathrm{CAN}(5, 10, 2) \le 56$; $\mathrm{CAN}(5, 9, 2) \le 54$; and $\mathrm{CAN}(5, 8, 2) \le 52$.

For strengths five and seven, the two sets of rows adjoined are complements, and hence there are pairs of disjoint rows. $\square$

3. Derived upper bounds

The following inequalities are basic ones that are used throughout; for most, the proofs are trivial and the results well known.

$$\text{reduction: } \mathrm{CAN}_r(s, n, q) \le \mathrm{CAN}_r(s + 1, n, q) \tag{1}$$

$$\text{truncation: } \mathrm{CAN}_r(s, n, q) \le \mathrm{CAN}_r(s, n + 1, q) \tag{2}$$

$$\text{composition: } \mathrm{CAN}(s, n, q_1 q_2) \le \mathrm{CAN}(s, n, q_1)\mathrm{CAN}(s, n, q_2). \tag{3}$$

Next we generalize a result from [13]:

Lemma 3.1 (*Fusion*).

$$\mathrm{CAN}_r(s,n,q) \le \mathrm{CAN}_r(s,n,q+1) - \begin{cases} 1 & \\ 2 & \text{if } r=0 \\ 3 & \text{if } r=0, s=2, n\le q+1, q \text{ a prime power}. \end{cases} \tag{4}$$

Proof. Permuting symbols within any column of a $\mathrm{CA}_r(M;s,n,q+1)$ produces another array with the same parameters. Applying permutations in each column, we can ensure that one row is constant, with every entry equal to $q+1$. Delete this row and change all other instances of $q+1$ in the array to any value in $\{1,\ldots,q\}$. The result is a $\mathrm{CA}_r(M-1;s,n,q)$.

When $r=0$, instead form the constant row of entry $q+1$ and delete it.

Now choose a second row R with entries $(\sigma_1,\ldots,\sigma_n)$. In all rows other than R, whenever an entry $q+1$ appears in column i, replace the entry by σ_i when $\sigma_i \le q$, or by any value in $\{1,\ldots,q\}$ otherwise. Then delete row R. We claim that the resulting array is a $\mathrm{CA}(M-2;s,n,q)$. To see this, consider any s-tuple of columns $(i_1,\ldots,i_s)$ for which none of $\{\sigma_{i_1},\ldots,\sigma_{i_s}\}$ is equal to $q+1$. The $\mathrm{CA}(M;s,n,q+1)$ permuted to contain a constant row must contain a row R' with $q+1$ in column i_1 and σ_{i_j} in column i_j for each $2\le j\le s$. Moreover, R' is neither the constant row nor the row R. Hence this s-tuple from row R is now covered in row R'. Thus row R covers no s-tuple not also covered in another row, and R can be deleted.

The last case, in which three rows are removed, is from [13]. □

$$\text{augmentation: } \mathrm{CAN}_r(s,n,q) \le \underbrace{\left\lfloor \frac{q}{q-1} \left\lfloor \frac{q}{q-1} \cdots \left\lfloor \frac{q\mathrm{CAN}_r(s,n,q-1)}{q-1} \right\rfloor \cdots \right\rfloor \right\rfloor}_{n \text{ times}}. \tag{5}$$

Inequality (5) results from adding a symbol to each column, one at a time, in a greedy manner. Let C_0 be a $\mathrm{CA}_r(M;s,n,q-1)$. Now for $i=1,\ldots,n$, we form C_i from C_{i-1} by first selecting a column γ in which only $q-1$ symbols occur. Then let σ' be a symbol not appearing in the column, and select the symbol σ appearing in this column that appears the least frequently. Then for every row that contains σ in column γ, we form a new row that is identical except that column γ contains σ' in place of σ. Chateauneuf and Kreher [9, Construction D] develop a different form of augmentation for the case when $s=3$:

$$\text{augmentation: } \mathrm{CAN}(3,n,q) \le \mathrm{CAN}(3,n,q-1) + n\mathrm{CAN}(2,n-1,q-1) + n(q-1). \tag{6}$$

Both augmentations operate by adding a new symbol, yet neither seems to be uniformly as good as the other. Inequality (5), although straightforward, does not seem to have been applied previously to bounding covering array numbers. Therefore in Table 1 we record improvements on [14] by augmentation.

We can reduce the strength:

$$\text{derivation: } \mathrm{CAN}(s,n,q) \le \left\lfloor \frac{\mathrm{CAN}(s+1,n+1,q)}{q} \right\rfloor. \tag{7}$$

One can also increase the number of columns:

Theorem 3.2.

$$\mathrm{CAN}(s+1,n+1,q) \le \mathrm{CAN}(s+1,n,q) + q(q-1)\mathrm{CAN}(s-1,n-1,q). \tag{8}$$

$$\mathrm{CAN}_r(s+1,n+1,q) \le \mathrm{CAN}_r(s+1,n,q) + (q-1)\mathrm{CAN}_r(s,n,q). \tag{9}$$

Proof. For (8), let A be a $\mathrm{CA}(M;s+1,n,q)$ and B be a $\mathrm{CA}(M';s-1,n-1,q)$. Form A' from A by duplicating the last column of A. For each $\{x,y\}$, form $B_{x,y}$ from B by adding two columns, the first containing x in each row, the second containing y. Form the $\mathrm{CA}(M+q(q-1)M';s+1,n+1,q)$ by vertically juxtaposing A' and $\{B_{x,y} : 0\le x,y<q, x\ne y\}$.

For (9), let A be a $\mathrm{CA}_r(M;s+1,n,q)$ and B be a $\mathrm{CA}_r(M';s,n,q)$. Form A' by adding a column that is constant, each entry being 0. Then for $1\le i<q$, form B_i by adding a column that is constant, each entry being i. Form the $\mathrm{CA}_r(M+(q-1)M';s+1,n+1,q)$ by vertically juxtaposing A' and $\{B_i : 1\le i<q\}$.

In both cases, the verification is routine. □

One should be able to improve Theorem 3.2 by choosing B in such a way that large parts of A are already covered, and can be removed. However, simply knowing the parameters of A and B does not ensure that they have any overlap at all. In one case, however, this can be easily done:

Lemma 3.3. *For q a prime power and $s\le q$,*

$$\mathrm{CAN}(s,q+2,q) \le \mathrm{CAN}(s,q+1,q) + q(q-1)\mathrm{CAN}(s-2,q,q) - q^{s-2}.$$

Table 1
Improvements found by applying augmentation.

n	CAN(4, n, q)						
	$q=10$	$q=12$	$q=14$	$q=18$	$q=21$	$q=22$	$q=24$
6		24677	44553				361253
7	13716	26920	47980	124611	262582		376959
8			51670				
9			55644				
10			59924				
11			64533				
n	CAN(5, n, q)						
	$q=10$	$q=12$	$q=14$	$q=18$	$q=21$	$q=22$	$q=24$
7		296130	623748			6194819	8670078
8	137173	323050	671728	2243020	5514335		9047037
9	152414	352418	723399	2374962	6094791		9440386
10			779045				
11			838971				
12			903507				
13			973007				
14			1047853				
n	CAN(6, n, q)						
	$q=10$	$q=12$	$q=14$	$q=15$	$q=18$	$q=21$	$q=24$
8	1234567	3553567	8732479	15165027	38131351	104772459	208081879
9	1371741	3876618	9404208		40374371	115801137	217128917
10	1524156	4229037	10127608		42749334	127990730	226569304
11		4613494	10906654		45264000	141463438	236420143
12			11745627				
13			12649136				
14			13622146				
n	CAN(s, n, 20)						
	$s=4$	$s=5$	$s=6$				
6	177285						
7	186615	3545706					
8	196436	3732322	70914127				
9	206774	3928760	74646449				
10	217656	4135536	78575209				
11	229111	4353195	82710746				
12	241169	4582310	87063943				
13	253862	4823484	91646255				
14	267223	5077351	96469742				
15		5344580	101547096				
16		5625873	106891680				
17		5921971	112517557				
18		6233653	118439533				
19			124673192				
20			131234938				

Proof. Now $\mathrm{CAN}(s, q+1, q) = q^s$ and $\mathrm{CAN}(s-2, q, q) = q^{s-2}$. Over the finite field $\mathbb{F}_q$, define an array A with columns indexed by the elements of $\mathbb{F}_q$ together with ∞, and rows indexed by polynomials of degree less than s over $\mathbb{F}_q$. In a row indexed by polynomial $p(x)$, place $p(i)$ in the entry in column i when i is an element of $\mathbb{F}_q$, and place the leading coefficient of $p(x)$ in the column indexed by ∞. The result is a $\mathrm{CA}(q^s; s, q+1, q)$ [22]. We partition the rows of A to form two arrays; A_1 contains rows indexed by polynomials of degree $s-2$ or $s-1$, and A_2 contains rows for polynomials of degree less than $s-2$. Then A_2 is a $\mathrm{CA}(q^{s-2}; s-2, q+1, q)$. Delete the column indexed by ∞ to obtain a $\mathrm{CA}(q^{s-2}; s-2, q, q)$, B. Now applying the proof of (8) in Theorem 3.2 to A and B, we obtain a $\mathrm{CA}(q^s+(q-1)q^{s-1}; s, q+2, q)$. But by construction, all rows of A_2 in A lead to rows that are redundant. Hence q^{s-2} rows can be removed to obtain the required $\mathrm{CA}(q^s+(q-1)q^{s-1}-q^{s-2}; s, q+2, q)$. □

Some additional inequalities are applicable only for radius-covering arrays with $r > 0$.

$$\mathrm{CAN}_r(s, n, q) \le \mathrm{CAN}_{r-1}(s-1, n-1, q). \tag{10}$$

Inequality (11) is Theorem 6 in [26]:

$$\mathrm{CAN}_{s+t-1-r}(s+t-1, n, p+q) \le \mathrm{CAN}_{s-r}(s, n, p) + \mathrm{CAN}_{t-r}(t, n, q). \tag{11}$$

4. Computational upper bounds

In the first two subsections, we focus on covering arrays.

$$A = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&2&2&2&1&1&2&0&2&0&0&2&1&1 \\ 2&0&2&0&1&2&0&1&0&1&1&1&0&2 \\ 1&0&1&2&2&2&1&1&0&0&0&2&0&0 \\ 2&0&0&2&0&0&1&1&2&1&2&1&2&1 \\ 0&0&1&1&0&2&1&2&2&2&1&0&1&2 \\ 1&2&0&0&0&2&1&2&1&0&0&1&2&2 \\ 0&0&1&2&1&0&0&2&1&0&1&2&2&1 \\ 2&2&1&0&2&0&1&0&0&2&1&1&2&1 \\ 1&2&0&2&1&0&2&1&1&2&1&0&0&2 \\ 0&2&2&0&1&0&1&1&2&0&0&2&1&2 \\ 0&0&1&2&2&0&2&0&1&1&0&1&1&2 \\ 2&1&1&1&2&0&0&1&2&0&0&0&2&2 \\ 0&1&0&0&2&2&2&1&2&0&1&1&0&1 \\ 0&2&1&0&1&2&2&1&1&1&2&0&2&0 \\ 0&2&0&2&2&2&0&0&2&1&1&2&2&2 \\ 0&2&1&2&0&0&1&0&2&2&2&2&0&0 \end{pmatrix}.$$

Fig. 1.

4.1. Upper bounds by cross-summing two codes

Let q be a positive integer. We typically interpret the binary operation '+' to be addition in the cyclic group $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$, but in some cases we instead employ addition in the elementary Abelian group arising (for example) from the finite field. In general, we can employ any group $\Gamma = (\{0, \ldots, q-1\}, +)$. The sum $a+b$ of two words $a, b \in \mathbb{Z}_q^n$ is defined as their coordinate-wise sum. Cross-summing of the codes $A \in \mathbb{Z}_q^n$ and $B \in \mathbb{Z}_q^n$ results in the code $C = \{a+b \mid a \in A, b \in B\}$, whose cardinality M is the product of the cardinalities M_A and M_B of A and B.

When addition is from additive group of a finite field, the cross-sum can also be interpreted as the union of suitable cosets of a linear code. Then the method is analogous to the *matrix method* from the theory of covering codes.

As an example, consider the codes $A, B \in \mathbb{Z}_3^{14}$ given in Fig. 1 in array form. A is a repetition code. The numbers of codewords in A and B are 3 and 17, respectively. In this case, and in all cases when A is a ternary repetition code, the cross-summing code C consists of the codewords of B, B' and B'', where B' and B'' are obtained from B and B' by the cyclic automorphism $0 \mapsto 1 \mapsto 2 \mapsto 0$. For this example, C is a 3-surjective ternary code, which proves the inequality $\mathrm{CAN}(3, 14, 3) \leq 51$, a significant improvement to the upper bound 60 implied by [15].

Good covering arrays can often be found by cross-summing of certain pairs of codes. We have had particular success when one of them is a repetition code (RC); the direct sum of repetition codes (DRC); or an extension of these with constant (all zero) coordinates (ERC or EDRC). The abbreviation ERC$a + b$ denotes the extension of an RC of length a by b constant coordinates. Sometimes the set of even words of a repetition code is used instead of the entire repetition code. Improvements are summarized in Table 2, where M is the improved bound on $\mathrm{CAN}(s, n, q)$, while M_{prev} is the previous best known upper bound given in [14]. The column 'Ref' gives an original reference.

For CAN(3, 14, 3), A and B are in Fig. 1. We give the exact array forms, A_7 and A_8, of A for two cases using direct sum of repetition codes.

$$A_7 = \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1 \\ 0&0&0&0&2&2&2 \\ 1&1&1&1&0&0&0 \\ 1&1&1&1&1&1&1 \\ 1&1&1&1&2&2&2 \\ 2&2&2&2&0&0&0 \\ 2&2&2&2&1&1&1 \\ 2&2&2&2&2&2&2 \end{pmatrix}; \qquad A_8 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&2&2&2 \\ 0&0&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&1 \\ 0&0&1&1&1&2&2&2 \\ 0&0&2&2&2&0&0&0 \\ 0&0&2&2&2&1&1&1 \\ 0&0&2&2&2&2&2&2 \end{pmatrix}.$$

For CAN(4, 6, 6), the direct sum of two senary repetition codes with three coordinates is used.

4.2. Upper bounds by simulated annealing

Simulated annealing proved to be an effective method of constructing good codes or arrays for many different purposes. The summary of the improvements for covering arrays that are new results and are obtained by using simulated annealing

Table 2
Improvements found by cross-summing.

s	n	q	$M = M_A \cdot M_B$	M_{prev}	Ref.	Method	File name
5	14	2	$64 = 2 \cdot 32$	103	[7,38]	ERC8 + 6	CA(32 × 2; 5, 14, 2)
5	15	2	$88 = 2 \cdot 44$	110	[7,38]	ERC9 + 6	CA(44 × 2; 5, 15, 2)
6	13	2	$128 = 2 \cdot 64$	205	[33,38]	ERC8 + 5	CA(64 × 2; 6, 13, 2)
6	15	2	$160 = 2 \cdot 80$	244	[33,38]	ERC8 + 7	CA(80 × 2; 6, 15, 2)
3	8	3	$42 = 3 \cdot 14$	45	[9]	RC	CA(14 × 3; 3, 8, 3)
3	12	3	$45 = 3 \cdot 15$	57	[9]	RC	CA(15 × 3; 3, 12, 3)
3	14	3	$51 = 3 \cdot 17$	60	[15]	RC	CA(17 × 3; 3, 14, 3)
3	15	3	$57 = 3 \cdot 19$	60	[15]	RC	CA(19 × 3; 3, 15, 3)
4	6	3	$111 = 3 \cdot 37$	115	[11]	RC	CA(37 × 3; 4, 6, 3)
4	7	3	$123 = 3 \cdot 41$	133	[11]	ERC6 + 1	CA(41 × 3; 4, 7, 3)
4	8	3	$141 = 3 \cdot 47$	153	[11]	ERC6 + 2	CA(47 × 3; 4, 8, 3)
4	11	3	$183 = 3 \cdot 61$	211	[33,38]	RC	CA(61 × 3; 4, 11, 3)
4	12	3	$201 = 3 \cdot 67$	237	[43]	RC	CA(67 × 3; 4, 12, 3)
4	13	3	$219 = 3 \cdot 73$	237	[43]	RC	CA(73 × 3; 4, 13, 3)
5	7	3	$351 = 9 \cdot 39$	377	[11]	DRC(A_7)	CA(39 × 9; 5, 7, 3)
6	8	3	$1152 = 9 \cdot 128$	1253	[38]	EDRC(A_8)	CA(128 × 9; 6, 8, 3)
3	7	5	$180 = 5 \cdot 36$	185	[9]	RC	CA(36 × 5; 3, 7, 5)
4	7	5	$910 = 5 \cdot 182$	1125	(8)	ERC5 + 2	CA(182 × 5; 4, 7, 5)
3	5	6	$240 = 3 \cdot 80$	260	[11]	RC	CA(80 × 3; 3, 5, 6)
3	6	6	$258 = 3 \cdot 86$	260	[11]	RC	CA(86 × 3; 3, 6, 6)
4	6	6	$1656 = 36 \cdot 46$	1728	[32]	DRC	CA(46 × 36; 4, 6, 6)

Table 3
Improvements found by simulated annealing.

s	n	q	M	M_{prev}	Ref	method	file name
6	10	2	116	142	[38]	SA	CA(116; 6, 10, 2)
2	12	4	24	26	[42]	SA	CA(24; 2, 12, 4)
2	13	4	25	26	[42]	SA	CA(25; 2, 13, 4)
4	6	4	340	375	[11]	SA	CA(340; 4, 6, 4)
2	12	5	38	39	[42]	SA	CA(38; 2, 12, 5)
2	13	5	40	41	[42]	SA	CA(40; 2, 13, 5)
3	7	6	293	314	[27]	SF	CA(293; 3, 7, 6)
3	8	6	304	342	(4)	SF	CA(304; 3, 8, 6)
3	9	6	379	423	[11]	DSF	CA(379; 3, 9, 6)
3	10	6	393	455	[9]	DSF	CA(393; 3, 10, 6)
4	7	6	1891	2380	[21]	SF	CA(1891; 4, 7, 6)
4	8	6	2044	2400	(4)	SF	CA(2044; 4, 8, 6)
2	9	7	59	61	[13,37]	SF	CA(59; 2, 9, 7)
2	13	7	76	77	[1]	SA	CA(76; 2, 13, 7)
3	9	7	472	510	(4)	SF	CA(472; 3, 9, 7)
3	10	7	479	510	(4)	SF	CA(479; 3, 10, 7)
2	7	10	113	118	[13]	SF	CA(113; 2, 7, 10)
2	8	10	115	118	[13]	SF	CA(115; 2, 8, 10)
2	11	10	116	118	[13]	SF	CA(116; 2, 11, 10)
2	12	10	117	118	[13]	SF	CA(117; 2, 12, 10)
2	13	11	156	161	[13,37]	DSF	CA(156; 2, 13, 11)
2	14	11	157	161	[13,37]	DSF	CA(157; 2, 14, 11)
2	8	12	162	166	[13]	SF	CA(162; 2, 8, 12)
2	9	12	163	166	[13]	SF	CA(163; 2, 9, 12)
2	12	12	164	166	[13]	SF	CA(164; 2, 12, 12)
2	14	12	165	166	[13]	SF	CA(165; 2, 14, 12)

is given in Table 3. The initial array was chosen by using either symbol fusion (SF), double symbol fusion (DSF), or without symbol fusion (abbreviated simply SA).

Complete listings of the arrays for Tables 2 and 3 are available at the web location http://www.sztaki.hu/~keri/arrays/CA_listings.zip. Their file names agree with the contents of the last column of each table.

4.3. Upper bounds for radius-covering arrays

Radius-covering arrays with radius at least one can also be produced by cross-summing and simulated annealing. We employ the results obtained in the tables of Section 8, but mention explicitly only those constructions that lead to strict equalities.

Theorem 4.1. $\mathrm{CAN}_1(6, 9, 2) = 16$, $\mathrm{CAN}_1(5, 11, 2) = 13$, $\mathrm{CAN}_2(7, 9, 2) = 10$, $\mathrm{CAN}_3(8, 12, 2) = 7$, $\mathrm{CAN}_1(4, 5, 3) = 14$, $\mathrm{CAN}_1(5, 6, 3) = 27$, $\mathrm{CAN}_4(8, 10, 3) = 9$, $\mathrm{CAN}_1(5, 6, 4) = 64$.

Table 4
Classification results for some binary covering arrays.

n	CA(12; 3, n, 2)	CA(14; 3, n, 2)	CA(24; 4, n, 2)
3	19	68	
4	79	657	1 981
5	33	1714	47 310
6	9	3376	434
7	2	3585	1
8	2	2395	1
9	1	1336	1
10	1	989	1
11	1	533	1
12	0	0	1
13			0

Table 5
Classification results for some ternary covering arrays.

n	CA(11; 2, n, 3)	CA(12; 2, n, 3)
2	3	7
3	20	134
4	27	987
5	3	891
6	0	13
7		1
8		0

All arrays found are available at the web location http://www.sztaki.hu/~keri/arrays/CAr_listings.zip.

5. Classification results and lower bounds

Classification entails finding all inequivalent solutions. Two solutions are *equivalent* when one is obtained by a row permutation, a column permutation, and independently chosen symbol permutations within each column. Classification results determine the number of solutions, and provide a main source of nonexistence results (when the number is 0). Our primary method for classification is an exhaustive computer search, in which the number of columns is increased stepwise from 2 to the desired number. To find the set of inequivalent CA(M; s, n, q)s, we start from the set of inequivalent CA(M; 1, 1, q)s, which can be obtained by a short computer program. Then we proceed as follows. The second and third arguments of CA(M; $\cdot$, $\cdot$, q) are incremented by 1 simultaneously until they reach the value of s, after which only the third argument of CA(M; s, $\cdot$, q) is incremented. For all q^M possible combinations of the new column a feasibility check and an equivalence check is then performed to build the set of inequivalent arrays.

In the first subsection, we focus on covering arrays.

5.1. Classification of covering arrays

Classification results for CA(M; 2, n, 2) appear in [29], in the terminology of surjective codes, for $6 \leq M \leq 8$ and arbitrary values of n. For $M = 5$, it can be proved by elementary combinatorial methods that CAN(2, 4, 2) = 5 and the corresponding CA(5; 2, 4, 2) is unique. In general, the uniqueness of CA(M; 2, n, 2) when $n = \binom{M-1}{\lfloor (M-2)/2 \rfloor}$ was proved by Katona [25]. Beyond this, Johnson and Entringer [24] establish that CAN($n - 2$, n, 2) $= \lfloor 2^n/3 \rfloor$ and that the corresponding covering array is unique.

For binary covering arrays where $2 < s < n - 2$ we give new classification results for $s \leq 4$ and $n \leq 12$. These are the numbers of inequivalent CA(12; 3, n, 2)s and CA(24; 4, n, 2)s in Table 4. Remarkably, each CA(24; 4, n, 2) with $n \in \{7, 8, \ldots, 12\}$ is unique.

The classification of CA(14; 3, n, 2)s yields an equality:

Theorem 5.1. CAN(3, 12, 2) = 15.

Proof. Nurmela [39] proves that CAN(3, 12, 2) $\leq$ 15, while Table 4 shows that no CA(14; 3, 12, 2) exists. $\square$

For ternary covering arrays we give new classification results for $s = 2$ and $n \leq 7$. The numbers of inequivalent CA(11; 2, n, 3)s and CA(12; 2, n, 3)s are shown in Table 5. Thus, we have three inequivalent for CA(11; 2, 5, 3)s, thirteen inequivalent CA(12; 2, 6, 3)s, and a unique CA(12; 2, 7, 3).

Theorem 5.2. CAN(2, 8, 3) = 13.

Proof. The inequality $\mathrm{CAN}(2, 8, 3) \leq 13$ is contained in [11]; the nonexistence of $\mathrm{CA}(12; 2, 8, 3)$ arrays follows from Table 5. $\square$

The complete listings of the covering arrays for which classification results are known can be found at the web location http://www.sztaki.hu/~keri/arrays/CCA_listings.zip.

5.2. Classification results and lower bounds for radius-covering arrays

Classification results for radius-covering arrays are also obtained by exhaustive computer search. For some simpler cases, the classification can be performed even without using a computer. The essentials of the classification results are contained in Table 6. The complete listings of the radius-covering arrays for which classification results exist can be found, for $r = 1, 2, 3$, at the web location http://www.sztaki.hu/~keri/arrays/CCAr_listings.zip.

6. Lower bounds and asymptotic formulas from irregularities in partitions

If equality occurs in $\mathrm{CAN}(s, n, q) \geq q \cdot \mathrm{CAN}(s-1, n-1, q)$ (i.e. the inequality (7)), then for a $\mathrm{CA}(s, n, q)$ with fewest rows, for each position there are precisely $\mathrm{CAN}(s-1, n-1, q)$ codewords having a given symbol at this position. Iterating this argument, if $\mathrm{CAN}(s, n, q) = q^d \mathrm{CAN}(s-d, n-d, q)$, every subarray with d columns has the property that every d-tuple is covered *exactly* $\mathrm{CAN}(s-d, n-d, q)$ times. In other words, the $\mathrm{CA}(s, n, q)$ is an orthogonal array of strength d and index $\mathrm{CAN}(s-d, n-d, q)$. This underlies a useful lower bound.

Theorem 6.1. *If* $\mathrm{CAN}(s-d, n-d, q) < q^{n-d}(1 - \frac{(q-1)n}{q(d+1)})$, *then* $\mathrm{CAN}(s, n, q) > q^d \mathrm{CAN}(s-d, n-d, q)$.

Proof. That $\mathrm{CAN}(s, n, q) \geq q^d \mathrm{CAN}(s-d, n-d, q)$ follows from (7). Suppose to the contrary that $\mathrm{CAN}(s-d, n-d, q) < q^{n-d}(1 - \frac{(q-1)n}{q(d+1)})$ and $\mathrm{CAN}(s, n, q) = q^d \mathrm{CAN}(s-d, n-d, q)$. Then a $\mathrm{CA}(\mathrm{CAN}(s, n, q); s, n, q)$ is an orthogonal array of strength d with $\mathrm{CAN}(s, n, q)$ rows. By [3, Theorem 1], it is necessary that $\mathrm{CAN}(s, n, q) \geq q^n \left(1 - \frac{(q-1)n}{q(d+1)}\right)$. Dividing both sides by q^d, we obtain that $\mathrm{CAN}(s-d, n-d, q) \geq q^{n-d}(1 - \frac{(q-1)n}{q(d+1)})$, which contradicts our assumption. $\square$

In a similar manner, other nonexistence results for orthogonal arrays may lead to lower bounds for covering array numbers.

When $d = 2$, a question arises: Is it possible to partition a finite set X into q subsets in many different ways, such that the intersection of any two sets occurring in different partitions is equal to $q^{-2}|X|$? We present two ways of dealing with this problem, the first one using linear algebra. An (n, M, q)-*partition matrix* is a $q \times n$-matrix with entries that are subsets of $[M]$, such that every column forms a partition of $[M]$.

Lemma 6.2. *Let M be an integer, divisible by q^{2k}. Let A be an (n, M, q)-partition matrix with $M < q^k \binom{n}{k}$. Then in A there exist $2k$ sets $A_1, \ldots, A_{2k}$ in different columns with intersection satisfying $|\bigcap A_i| < M/q^{2k}$.*

Proof. Suppose otherwise. Then for $\ell \leq 2k$ the intersection of ℓ sets in different columns has size exactly M/q^ℓ. For sets $A_1, \ldots, A_\ell$ in different columns define the vector $v(A_1, \ldots, A_\ell) \in \mathbb{R}^M$ as the vector having entry 1 at coordinate i if $i \in \bigcap_{i=1}^{\ell} A_i$, and 0 otherwise. Using the assumption on the intersection of the sets in A we can compute the scalar product of two such vectors. Let $c_1, \ldots, c_n$ be the columns of A, and write $A_1 \sim A_2$, if the sets A_1, A_2 occur in the same column of A. Let ℓ, ℓ' be integers with $\ell + \ell' \leq 2k$. Choose sets $A_1, \ldots, A_\ell, A'_1, \ldots, A'_{\ell'}$. If there are sets A_{i_0}, A'_{j_0} with $A_{i_0} \sim A'_{j_0}$, but $A_{i_0} \neq A'_{j_0}$, then $\bigcap A_i \cap \bigcap A'_j \subseteq A_{i_0} \cap A'_{j_0} = \emptyset$, and the product is 0. If there are no such indices, then $\left|\bigcap A_i \cap \bigcap A'_j\right|$ equals $Mq^{-\nu}$, where ν is the number of different sets among $A_1, \ldots, A_\ell, A'_1, \ldots, A'_{\ell'}$. In this case sets are equal if and only if they are in the same column, that is, ν equals $2k$ minus the number m of indices $i_1, \ldots, i_m$, for which there are indices $j_1, \ldots, j_m$ such that $A_{i_\nu} \sim A'_{j_\nu}$, that is,

$$\langle v(A_1, \ldots, A_\ell), v(A'_1, \ldots, A'_\ell) \rangle = \begin{cases} 0, & \exists i, j : A_i \sim A'_j, \text{ but } A_i \neq A'_j \\ Mq^{m-\ell-\ell'}, & \text{There are precisely } m \text{ indices with } i_1, \ldots, i_m, j_1, j_m \text{ with } A_{i_\mu} = A'_{j_\mu}. \end{cases}$$

There are $q^k \binom{n}{k}$ vectors of the form $v(A_1, \ldots, A_k)$. Because M is smaller, these vectors are linearly dependent; that is, there exist real numbers $\lambda(A_1, \ldots, A_k)$, not all 0, such that

$$\sum_{A_1, \ldots, A_k} \lambda(A_1, \ldots, A_k) v(A_1, \ldots, A_k) = 0. \tag{12}$$

Here and in the sequel we always sum over sets of sets in different columns. For sets $A_1, \ldots, A_\ell$ in different columns, define

$$S(A_1, \ldots, A_\ell) = \sum_{\substack{A'_1, \ldots, A'_k \\ \{A_1, \ldots, A_\ell\} \subseteq \{A'_1, \ldots, A'_k\}}} \lambda(A_1, \ldots, A_k).$$

Table 6
Classification results for radius-covering arrays.

n	$CA_1(4; 4, n, 2)$	$CA_1(5; 4, n, 2)$	$CA_1(6; 4, n, 2)$	$CA_1(7; 4, n, 2)$
4	2	7	45	160
5	1	6	65	446
6	0	1	33	597
7		0	10	515
8			1	211
9			0	33
10				0

n	$CA_1(7; 5, n, 2)$	$CA_1(8; 5, n, 2)$	$CA_1(10; 5, n, 2)$	$CA_1(11; 5, n, 2)$	$CA_1(12; 5, n, 2)$
5	1	34	3178	23414	148090
6	1	22	4952	76610	1084818
7	0	1	65	2337	199890
8		0	3	141	17649
9			0	2	395
10				0	17
11					0

n	$CA_1(12; 6, n, 2)$	$CA_1(16; 7, n, 2)$	$CA_2(7; 7, n, 2)$	$CA_2(12; 8, n, 2)$	$CA_2(16; 9, n, 2)$
6	2				
7	1	1	3		
8	0	1	1	277	
9		0	0	48	4
10				0	2
11					0

n	$CA_2(4; 6, n, 2)$	$CA_2(5; 6, n, 2)$	$CA_2(6; 6, n, 2)$	$CA_2(7; 6, n, 2)$
6	4	23	420	5354
7	2	16	404	9439
8	0	2	105	5535
9		0	23	2464
10			0	457
11				10
12				0

n	$CA_3(4; 8, n, 2)$	$CA_3(5; 8, n, 2)$	$CA_3(6; 8, n, 2)$	$CA_3(7; 9, n, 2)$
8	6	59	2525	
9	3	36	1846	8
10	0	3	279	3
11		0	42	0
12			0	

n	$CA_1(6; 3, n, 3)$	$CA_1(7; 3, n, 3)$	$CA_2(9; 5, n, 3)$
3	10	99	
4	7	213	
5	0	89	518
6		28	7
7		4	2
8		1	1
9		1	0
10		1	
11		0	

n	$CA_1(10; 3, n, 4)$	$CA_1(11; 3, n, 4)$	$CA_1(14; 3, n, 5)$	$CA_2(8; 4, n, 4)$
3	49	784	7	
4	8	500	1	1540
5	0	7	0	448
6		0		69
7				11

We claim that for $\ell \le k$ and every choice of the sets $A_1, \dots, A_\ell$ in different columns, $S(A_1, \dots, A_\ell) = 0$. A proof of this claim suffices to prove the theorem, because $S(A_1, \dots, A_k) = \lambda(A_1, \dots, A_k)$ so that the coefficients in the linear combination vanish, which is a contradiction.

We prove the claim by induction on ℓ. For $\ell = 0$ write $S(\emptyset) = \sum_{A_1,\dots,A_k} \lambda(A_1, \dots, A_k)$, and $v(\emptyset)$ is the vector having 1 at each coordinate. Taking the scalar product of (12) with $v(\emptyset)$, we establish the claim for $\ell = 0$. Now suppose that the claim is true for all $\lambda \le \ell - 1$; we prove it for ℓ. Choose sets $A_1, \dots, A_\ell$. Our aim is to show that for $\ell \le k$, $S(A_1, \dots, A_\ell) = 0$. To do so take the scalar product of (12) with $v(A_1, \dots, A_\ell)$. To compute the scalar product with one summand $\lambda(A'_1, \ldots, A'_k)v(A'_1, \ldots, A'_k)$, we examine whether there exist indices i, j such that $A_i \sim A'_j$, but $A_i \neq A_j$. When this does not hold, we must compute the size of $\{A_1, \ldots, A_\ell\} \cap \{A'_1, \ldots, A'_k\}$. Let $c_1, \ldots, c_\ell$ be the columns of A containing $A_1, \ldots, A_\ell$, respectively. Then

$$\sum_{A'_1,\ldots,A'_k} \lambda(A'_1, \ldots, A'_k)\langle v(A_1, \ldots, A_\ell), v(A'_1, \ldots, A'_k)\rangle = \sum_{I\subset[\ell]} \frac{M}{q^{\ell+k-|I|}} \sum_{\substack{A'_1,\ldots,A'_k \\ (\exists j: A'_j \in c_i)\Leftrightarrow i\in I \\ \forall i,j: A_i\sim A'_j \Rightarrow A_i = A'_j}} \lambda(A'_1, \ldots, A'_k)$$

$$= \sum_{I\subset[\ell]} \frac{M}{q^{\ell+k-|I|}} S(\{A_i, i \in I\}) = \frac{M}{q^k} S(A_1, \ldots, A_\ell),$$

because by the inductive hypothesis $S(\{A_i, i \in I\}) = 0$ for every proper subset I of $[\ell]$. However by (12) the left hand side vanishes. Hence $S(A_1, \ldots, A_\ell) = 0$, and the claim is proved. □

Corollary 6.3. $\mathrm{CAN}(6, 10, 2) \geq 112$.

We also obtain $\mathrm{CAN}(4, 13, 2) \geq 26$ and $\mathrm{CAN}(4, 14, 2) \geq 28$ from Lemma 6.2, but these are improved by the classification results and (7).

The second approach uses bounds on error correcting codes to deduce certain irregularities. Unfortunately, it appears that this approach only gives non-trivial results for $q = 2$.

Lemma 6.4. *Let M be an integer, and A an $(M, n, 2)$-partition matrix. Set $m = \min |A_{ij} \cap A_{i'j'}|$, where the minimum is taken over all quadruples (i, j, i', j') with $j \neq j'$, and suppose that $m > 0$. Then there exists a code $C \subseteq \mathbb{Z}_2^{M-1}$ with minimal distance $2m$ and size $2n$.*

Proof. For $1 \leq j \leq n$ define a codeword $c \in \mathbb{Z}_2^M$ having 1 at position i, if in the jth column of A the elements 1 and i are in the same set, and 0 otherwise. In this way all codewords begin with 1; deleting the first position yields a code $C' \subseteq \mathbb{Z}_2^{M-1}$ of size n. Next let C'' be C' with the digits 0 and 1 interchanged. Because $m > 0$, C' and C'' are disjoint, and hence $C = C' \cup C''$ is a code of size $2n$. We claim that C has minimal distance $2m$. Suppose this is not the case, and let c_1, c_2 be codewords with $d(c_1, c_2) \leq 2m - 1$. Then without loss of generality we assume that there are at most $m - 1$ positions at which c_1 has the digit 0, and c_2 has the digit 1. But then the intersection of the set in A corresponding to the digit 0 in the column corresponding to c_1 and the set corresponding to 1 in the column corresponding to c_2 has at most $m - 1$ elements, which gives a contradiction. □

To bound the size of linear codes we employ a result of McEliece, Rodemich, Rumsey, and Welch [36]:

Lemma 6.5. *Set $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$. Let $C \subseteq \mathbb{Z}_2^n$ be a code with minimum distance d. Then as $n \to \infty$*

$$\frac{1}{n} \log_2 |C| \leq (1 + o(1))H_2\left(\frac{1}{2} - \sqrt{\frac{d}{n}\left(1 - \frac{d}{n}\right)}\right).$$

Corollary 6.6. *For n sufficiently large,* $\mathrm{CAN}(4, n, 2) \geq 5.84 \log_2 n$.

Proof. We have $\mathrm{CAN}(2, n, 2) \sim \log_2 n$. Now let C be a binary code of length n. Select two positions i_1, i_2 and two symbols e_1, e_2. If there are fewer than $\mathrm{CAN}(2, n - 2, 2)$ codewords in C having symbol e_j in position i_j for $j \in \{1, 2\}$, C cannot be 4-surjective with radius 2. Hence there exists a binary code of length $|C| - 1$, size n, and minimal distance at least $(2 + o(1)) \log_2 n$. The claim follows from solving the equation

$$\frac{\log_2 n}{|C|} = H_2\left(\frac{1}{2} - \sqrt{\frac{2\log n}{|C|}\left(1 - \frac{2\log n}{|C|}\right)}\right)$$

numerically. We obtain $\mathrm{CAN}(4, n, 2) \geq (5.8401 \cdots + o(1)) \log_2 n$, which for n large implies the claim. □

7. Exact values and upper bounds from covering codes

Here we use only uniform covering codes, which are arbitrary non-empty subsets of Z_q^n, the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $Z_q = \{0, 1, \ldots, q - 1\}$. The *Hamming distance* $d(x, y)$ between two words $x, y \in Z_q^n$ is the number

Table 7
Key to the tables of Section 8.

Lower bounds							
α	[4,6,25,30]	β	[24]	ω	[47]	a	OA
b	Theorem 6.1	d	(2)	e	(1)	f	Theorem 8.2
g	Covering code	h	[41]	j	[26]	k	[29]
p	(17)	r	(7)	t	Classification Section 5	v	(18)
x	[20]	y	Computational	z	Linear algebra Section 6		

Upper bounds							
α	[4,6,25,30]	β	[24]	γ	[16]	δ	[44]
ϵ	[32]	ζ	[9]	η	[49]	θ	[13,37]
κ	[39]	λ	[42]	μ	[11]	ν	[48]
ξ	[23]	π	[38]	ρ	[33,38]	σ	[7,38]
τ	[18,38]	ϕ	[50]	ψ	[34]	a	OA
c	(5)	d	(2)	e	(1)	f	(10)
g	Covering code	h	[41]	j	[26]	m	(3)
n	(11)	p	Lemma 2.1	q	(4)	r	(7)
s	New (Section 4)	t	Classification Section 5	u	Lemma 3.3	x	Theorem 7.3
y	(13)–(16)	z	(8) or (9)				

$\mathrm{CAN}_r(s, n, 2)$

$n, s \setminus r$	0	1	2	3	4	5
2, 2	a $\mathbf{4}^{1}$ a	$\mathbf{2}^{2}$				
3, 2	a $\mathbf{4}^{1}$ a	2				
3, 3	a $\mathbf{8}^{1}$ a	$\mathbf{2}^{1}$				
4, 2	α $\mathbf{5}^{1}$ α	2				
4, 3	a $\mathbf{8}^{1}$ a	$\mathbf{2}^{1}$				
4, 4	a $\mathbf{16}^{1}$ a	g $\mathbf{4}^{2}$ g	$\mathbf{2}^{2}$			
5, 2	α $\mathbf{6}^{7}$ α	2				
5, 3	β $\mathbf{10}^{1}$ β	$\mathbf{2}^{1}$				
5, 4	a $\mathbf{16}^{1}$ a	d $\mathbf{4}^{1}$ y	2			
5, 5	a $\mathbf{32}^{1}$ a	g $\mathbf{7}^{1}$ g	$\mathbf{2}^{1}$			
6, 2	α $\mathbf{6}^{4}$ α	2				
6, 3	r $\mathbf{12}^{9}$ d	$\mathbf{2}^{1}$				
6, 4	β $\mathbf{21}^{1}$ β	h $\mathbf{5}^{1}$ h	2			
6, 5	a $\mathbf{32}^{1}$ a	d $\mathbf{7}^{1}$ y	$\mathbf{2}^{1}$			
6, 6	a $\mathbf{64}^{1}$ a	g $\mathbf{12}^{2}$ g	g $\mathbf{4}^{4}$ g	$\mathbf{2}^{2}$		
7, 2	α $\mathbf{6}^{3}$ α	2				
7, 3	d $\mathbf{12}^{2}$ d	$\mathbf{2}^{1}$				
7, 4	r $\mathbf{24}^{1}$ d	h $\mathbf{6}^{10}$ h	2			
7, 5	β $\mathbf{42}^{1}$ β	h $\mathbf{8}^{1}$ h	$\mathbf{2}^{1}$			
7, 6	a $\mathbf{64}^{1}$ a	d $\mathbf{12}^{1}$ y	d $\mathbf{4}^{2}$ y	2		
7, 7	a $\mathbf{128}^{1}$ a	g $\mathbf{16}^{1}$ g	g $\mathbf{7}^{3}$ g	$\mathbf{2}^{1}$		
8, 2	α $\mathbf{6}^{1}$ α	2				
8, 3	d $\mathbf{12}^{2}$ d	$\mathbf{2}^{1}$				
8, 4	d $\mathbf{24}^{1}$ d	d $\mathbf{6}^{1}$ t	2			
8, 5	r 48–52 p	t $\mathbf{10}^{3}$ t	$\mathbf{2}^{1}$			
8, 6	β $\mathbf{85}^{1}$ β	y 16 e	y $\mathbf{5}^{2}$ t	2		
8, 7	a $\mathbf{128}^{1}$ a	e $\mathbf{16}^{1}$ y	d $\mathbf{7}^{1}$ y	$\mathbf{2}^{1}$		
8, 8	a $\mathbf{256}^{1}$ a	g $\mathbf{32}^{10}$ g	g $\mathbf{12}^{277}$ g	g $\mathbf{4}^{6}$ g	$\mathbf{2}^{2}$	
9, 2	α $\mathbf{6}^{1}$ α	2				
9, 3	d $\mathbf{12}^{1}$ d	$\mathbf{2}^{1}$				
9, 4	d $\mathbf{24}^{1}$ d	h $\mathbf{7}^{33}$ t	2			
9, 5	d 48–54 p	t $\mathbf{11}^{2}$ t	$\mathbf{2}^{1}$			
9, 6	r 96–111 p	d 16 s	y $\mathbf{6}^{23}$ t	2		
9, 7	β $\mathbf{170}^{1}$ β	y 21–24 d	y 10 s	$\mathbf{2}^{1}$		
9, 8	a $\mathbf{256}^{1}$ a	d 32 x	d $\mathbf{12}^{48}$ y	d $\mathbf{4}^{3}$ y	2	
9, 9	a $\mathbf{512}^{1}$ a	g 62 g	g $\mathbf{16}^{4}$ g	g $\mathbf{7}^{8}$ g	$\mathbf{2}^{1}$	

(*continued on next page*)

of coordinates in which they differ. Extending this definition, $d(x, C) = \min(d(x, y) : y \in C)$. The *covering radius* of a code $C \subseteq Z_q^n$ is $R = \max\{d(x, C) \mid x \in Z_q^n\}$.

Let $K_q(n, R)$ denote the minimum number of codewords in a q-ary code with n coordinates and covering radius R. An easy proof by the pigeonhole principle establishes:

Table 7 (*continued*)

10, 2	$\alpha\ \mathbf{6}^{1}\alpha$	2				
10, 3	d $\mathbf{12}^{1}$ d	$\mathbf{2}^{1}$				
10, 4	d $\mathbf{24}^{1}$ d	y 8 n	2			
10, 5	d 48–56 p	t $\mathbf{12}^{17}$ t	$\mathbf{2}^{1}$			
10, 6	z 112–116 s	d 16–20 d	y $\mathbf{7}^{457}$ d	2		
10, 7	r 192–222 p	d 21–24 d	d 10–11 s	$\mathbf{2}^{1}$		
10, 8	$\beta\ \mathbf{341}^{1}\beta$	d 32–56 z	y 13–16 e	t $\mathbf{5}^{3}$ t	2	
10, 9	a $\mathbf{512}^{1}$ a	d 62 x	d $\mathbf{16}^{2}$ t	d $\mathbf{7}^{3}$ y	$\mathbf{2}^{1}$	
10, 10	a $\mathbf{1024}^{1}$ a	g 107–120 g	g 24–30 g	g $\mathbf{12}^{11481}$ g	g $\mathbf{4}^{9}$ g	$\mathbf{2}^{2}$

$\mathrm{CAN}_r(s, n, 2)$

$n, s \setminus r$	0	1	2	3	4	5	6
11, 2	$\alpha\ \mathbf{7}^{26}\alpha$	2					
11, 3	d $\mathbf{12}^{1}$ r	$\mathbf{2}^{1}$					
11, 4	d $\mathbf{24}^{1}$ d	d 8 n	2				
11, 5	d 48–64 d	y 13 s	$\mathbf{2}^{1}$				
11, 6	d 112–128 d	d 16–20 s	d $\mathbf{7}^{10}$ t	2			
11, 7	r 224–242 p	d 21–24 d	d 10–12 s	$\mathbf{2}^{1}$			
11, 8	b 385–563 z	d 32–80 z	d 13–16 s	t $\mathbf{6}^{42}$ t	2		
11, 9	$\beta\ \mathbf{682}^{1}\beta$	d 62–118 z	y 17–24 s	y 9–10 s	$\mathbf{2}^{1}$		
11, 10	a $\mathbf{1024}^{1}$ a	d 107–120 x	d 24–30 x	d 12 y	d 4 y	2	
11, 11	a $\mathbf{2048}^{1}$ a	g 180–192 g	g 37–44 g	g 15–16 g	g $\mathbf{7}^{17}$ g	$\mathbf{2}^{1}$	
12, 2	$\alpha\ \mathbf{7}^{10}\alpha$	2					
12, 3	y 15 κ	$\mathbf{2}^{1}$					
12, 4	d $\mathbf{24}^{1}\gamma$	d 8 n	2				
12, 5	d 48–64 d	d 13–21 z	$\mathbf{2}^{1}$				
12, 6	d 112–128 d	d 16–24 e	y 8 n	2			
12, 7	d 224–264 p	d 21–24 s	d 10–19 z	$\mathbf{2}^{1}$			
12, 8	r 448–795 z	d 32–104 z	d 13–24 f	y 7 s	2		
12, 9	r 770–1230 z	d 62–192 e	d 17–40 z	d 9–16 e	$\mathbf{2}^{1}$		
12, 10	$\beta\ \mathbf{1365}^{1}\beta$	d 107–192 e	d 24–44 e	d 12–16 e	d 4–6 z	2	
12, 11	a $\mathbf{2048}^{1}$ a	d 180–192 x	d 37–44 x	d 15–16 y	d 7 y	$\mathbf{2}^{1}$	
12, 12	a $\mathbf{4096}^{1}$ a	g 342–380 g	g 62–78 g	g 18–28 g	g 11–12 g	g $\mathbf{4}^{12}$ g	$\mathbf{2}^{2}$
13, 2	$\alpha\ \mathbf{7}^{4}\alpha$	2					
13, 3	d 15–16 δ	$\mathbf{2}^{1}$					
13, 4	r 30–32 ν	d 8 n	2				
13, 5	d 48–64 d	d 13–24 f	$\mathbf{2}^{1}$				
13, 6	d 112–128 s	d 16–45 z	d 8 n	2			
13, 7	d 224–392 z	d 21–48 z	d 10–24 f	$\mathbf{2}^{1}$			
13, 8	d 448–1051 z	d 32–128 z	d 13–24 f	d 7–9 z	2		
13, 9	r 896–2002 z	d 62–296 z	d 17–64 z	d 9–23 z	$\mathbf{2}^{1}$		
13, 10	r 1540–2491 z	d 107–380 e	d 24–78 e	d 12–28 e	d 4–8 z	2	
13, 11	$\beta\ \mathbf{2730}^{1}\beta$	d 180–380 e	d 37–78 e	d 15–28 e	d 7–12 e	$\mathbf{2}^{1}$	
13, 12	a $\mathbf{4096}^{1}$ a	d 342–380 x	d 62–78 x	d 18–28 x	d 11–12 y	d 4 y	2
13, 13	a $\mathbf{8192}^{1}$ a	g 598–704 g	g 97–128 g	g 28–42 g	y 12–16 g	g $\mathbf{7}^{33}$ g	$\mathbf{2}^{1}$
14, 2	$\alpha\ \mathbf{7}^{1}\alpha$	2					
14, 3	d 15–16 δ	$\mathbf{2}^{1}$					
14, 4	d 30–35 ν	d 8 n	2				
14, 5	r 60–64 s	d 13–32 f	$\mathbf{2}^{1}$				
14, 6	d 112–160 s	d 16–64 f	d 8 n	2			
14, 7	d 224–520 z	d 21–93 z	d 10–32 z	$\mathbf{2}^{1}$			
14, 8	d 448–1307 z	d 32–176 z	d 13–48 z	f 8–11 z	2		
14, 9	d 896–3014 z	d 62–424 z	d 17–88 z	d 9–24 f	$\mathbf{2}^{1}$		
14, 10	r 1792–4081 z	d 107–676 z	d 24–128 e	d 12–42 e	d 4–10 z	2	
14, 11	r 3080–5190 z	d 180–704 e	d 37–128 e	d 15–42 e	d 7–16 e	$\mathbf{2}^{1}$	
14, 12	$\beta\ \mathbf{5461}^{1}\beta$	d 342–704 e	d 62–128 e	d 18–42 e	d 11–16 e	d 4–6 z	2
14, 13	a $\mathbf{8192}^{1}$ a	d 598–704 x	d 97–128 x	d 28–42 x	d 12–16 y	d 7 y	$\mathbf{2}^{1}$
14, 14	a $\mathbf{16384}^{1}$ a	g 1172–1408 g	g 159–248 g	g 44–64 g	g 16–28 g	g 10–12 g	g $\mathbf{4}^{16}$ g
15, 2	$\alpha\ \mathbf{7}^{1}\alpha$	2					

(*continued on next page*)

Proposition 7.1. *$K_q(s, r) = q$ if and only if $r < s < \frac{qr+q}{q-1}$, and an optimal covering code belonging to this case is the repetition code in Z_q^s.*

Table 7 (*continued*)

$\mathrm{CAN}_r(s, n, 3)$							
$n, s \setminus r$	0	1	2	3	4	5	6
2, 2	a $\mathbf{9}^1$ a	$\mathbf{3}^3$					
3, 2	a $\mathbf{9}^1$ a	3					
3, 3	a $\mathbf{27}^1$ a	g $\mathbf{5}^1$ g	$\mathbf{3}^7$				
4, 2	a $\mathbf{9}^1$ a	3					
4, 3	a 27 a	j $\mathbf{6}^7$ j	3				
4, 4	a $\mathbf{81}^1$ a	g $\mathbf{9}^1$ g	$\mathbf{3}^1$				
5, 2	t $\mathbf{11}^3$ t	3					
5, 3	v 33 d	k $\mathbf{7}^{89}$ n	3				
5, 4	a 81 a	y 14 s	$\mathbf{3}^1$				
5, 5	a $\mathbf{243}^1$ a	g $\mathbf{27}^{17}$ g	g $\mathbf{8}^1$ g	$\mathbf{3}^3$			
6, 2	t $\mathbf{12}^{13}$ t	3					
6, 3	d 33 ζ	k $\mathbf{7}^{28}$ n	3				
6, 4	r 99–111 s	d 14–17 s	$\mathbf{3}^1$				
6, 5	a 243 a	d 27 s	t $\mathbf{9}^7$ t	3			
6, 6	a $\mathbf{729}^1$ a	g 71–73 g	g 15–17 g	g $\mathbf{6}^{28}$ g	$\mathbf{3}^7$		
7, 2	t $\mathbf{12}^1$ t	3					
7, 3	r 36–40 ξ	k $\mathbf{7}^4$ n	3				
7, 4	d 99–123 s	d 14–18 s	$\mathbf{3}^1$				
7, 5	r 297–351 s	d 27–61 z	d $\mathbf{9}^2$ t	3			
7, 6	a 729 a	d 71–127 z	d 15–21 s	d $\mathbf{6}^7$ t	3		
7, 7	a $\mathbf{2187}^1$ a	g 156–186 g	g 26–34 g	g 11–12 g	$\mathbf{3}^1$		
8, 2	y 13 d	3					
8, 3	d 36–42 s	k $\mathbf{7}^1$ n	3				
8, 4	r 108–141 s	d 14–32 z	$\mathbf{3}^1$				
8, 5	d 297–432 ϕ	d 27–97 z	d $\mathbf{9}^1$ t	3			
8, 6	r 891–1152 s	d 71–249 z	d 15–39 z	t $\mathbf{7}^{71}$ t	3		
8, 7	a 2187 a	d 156–406 c	d 26–76 z	d 11–15 s	$\mathbf{3}^1$		
8, 8	a $\mathbf{6561}^1$ a	g 402–486 g	g 54–81 g	g 14–27 g	g 9 g	$\mathbf{3}^3$	
9, 2	d 13 μ	3					
9, 3	r 39–45 d	k $\mathbf{7}^1$ n	3				
9, 4	d 108–159 d	d 14–42 f	$\mathbf{3}^1$				
9, 5	r 324–483 d	d 27–141 f	y 10–12 s	3			
9, 6	d 891–1449 d	d 71–432 f	d 15–57 z	t $\mathbf{7}^5$ t	3		
9, 7	r 2673–4293 z	d 156–904 z	d 26–154 z	d 11–25 n	$\mathbf{3}^1$		
9, 8	a 6561 a	d 402–1228 c	d 54–219 e	d 14–27 d	d 9 d	3	
9, 9	a $\mathbf{19683}^1$ a	g 1060–1269 g	g 130–219 g	g 27–54 g	y 11–18 g	g 6 g	$\mathbf{3}^7$
10, 2	d 13–14 μ	3					
10, 3	d 39–45 s	k $\mathbf{7}^1$ n	3				
10, 4	r 117–159 η	d 14–45 f	$\mathbf{3}^1$				
10, 5	d 324–483 r	d 27–159 f	d 10–13 n	3			
10, 6	r 972–1449 η	d 71–483 f	d 15–81 z	y 8–9 n	3		
10, 7	d 2673–6885 z	d 156–1369 c	d 26–268 z	d 11–25 n	$\mathbf{3}^1$		
10, 8	r 8019–13473 z	d 402–3036 z	d 54–527 z	d 14–27 s	d 9 s	3	
10, 9	a 19683 a	d 1060–3552 c	d 130–555 e	d 27–105 e	d 11–27 f	d 6–8 n	3
10, 10	a $\mathbf{59049}^1$ a	g 2854–3645 g	g 323–555 g	g 57–105 g	g 17–36 g	g 9–12 g	$\mathbf{3}^1$

(*continued on next page*)

Taking the repetition code in Z_q^n for $n \geq s$:

Corollary 7.2. $\mathrm{CAN}_r(s, n, q) = q$ *if and only if* $r < s < \frac{qr+q}{q-1}$.

Hence $\mathrm{CAN}_r(s, n, q)$ is interesting primarily when $s \geq \left\lceil \frac{qr+q}{q-1} \right\rceil$.

For binary radius-covering arrays when $n = s + 1$ the following equality holds:

Theorem 7.3. $\mathrm{CAN}_r(s, s+1, 2) = \mathrm{CAN}_r(s, s, 2) = K_2(s, r)$.

Proof. Let C be a binary covering code corresponding to $K(s, r) = M$. By extending C with an additional coordinate, containing the parity checksums for each codeword of C, a $\mathrm{CA}_r(M; s, s+1, 2)$ is obtained. To show this, it is enough to prove that replacing the last coordinate of the codewords of C with the checksum bits, the resulting new code D has covering

Table 7 (*continued*)

$\mathrm{CAN}_r(s, n, 4)$						
$n, s \setminus r$	0	1	2	3	4	5
2, 2	a $\mathbf{16}^{1}$ a	$\mathbf{4}^{5}$				
3, 2	a $\mathbf{16}^{2}$ a	4				
3, 3	a $\mathbf{64}^{1}$ a	g $\mathbf{8}^{1}$ g	$\mathbf{4}^{21}$			
4, 2	a $\mathbf{16}^{1}$ a	4				
4, 3	a 64 a	h $\mathbf{10}^{8}$ n	4			
4, 4	a $\mathbf{256}^{1}$ a	g 24 g	g $\mathbf{7}^{8}$ g	$\mathbf{4}^{79}$		
5, 2	a $\mathbf{16}^{1}$ a	4				
5, 3	a 64 a	h $\mathbf{11}^{7}$ h	4			
5, 4	a 256 a	d 24–32 s	j $\mathbf{8}^{448}$ j	4		
5, 5	a $\mathbf{1024}^{1}$ a	g 64 g	x 16 g	$\mathbf{4}^{1}$		
6, 2	p 19 θ	4				
6, 3	a 64 a	h 12 n	4			
6, 4	p 257–340 s	d 24–64 e	d $\mathbf{8}^{69}$ n	4		
6, 5	a 1024 a	d 64 s	d 16–24 n	$\mathbf{4}^{1}$		
6, 6	a $\mathbf{4096}^{1}$ a	g 228–256 g	g 32–52 g	x 11–14 g	$\mathbf{4}^{5}$	
7, 2	d 19–21 μ	4				
7, 3	r 76–88 d	h 12 n	4			
7, 4	d 257–450 π	d 24–64 f	d $\mathbf{8}^{11}$ n	4		
7, 5	r 1028–1536 ϵ	d 64–256 e	d 16–24 n	$\mathbf{4}^{1}$		
7, 6	a 4096 a	d 228–256 s	d 32–64 f	d 11–16 h	4	
7, 7	a $\mathbf{16384}^{1}$ a	g 762–992 g	g 84–128 g	g 19–32 g	x 9–10 g	$\mathbf{4}^{21}$
8, 2	d 19–22 μ	4				
8, 3	d 76–88 ζ	d 12 n	4			
8, 4	r 304–508 d	d 24–88 f	d 8 n	4		
8, 5	d 1028–2044 d	d 64–448 z	d 16–24 n	$\mathbf{4}^{1}$		
8, 6	r 4112–7225 m	d 228–1024 z	d 32–136 z	d 11–18 n	4	
8, 7	a 16384 a	d 762–1760 z	d 84–256 f	d 19–32 s	d 9–10 n	4
8, 8	a $\mathbf{65536}^{1}$ a	g 2731–3456 g	g 240–352 g	g 44–96 g	g 13–28 g	g 8 g
9, 2	d 19–23 λ	4				
9, 3	d 76–112 d	d 12 n	4			
9, 4	d 304–508 d	d 24–88 f	d 8 n	4		
9, 5	r 1216–2044 d	d 64–508 f	d 16–24 n	$\mathbf{4}^{1}$		
9, 6	d 4112–9191 ρ	d 228–2044 f	d 32–208 z	f 12–19 n	4	
9, 7	r 16448–28900 m	d 762–4832 z	d 84–664 z	d 19–48 n	d 9–13 n	4
9, 8	a 65536 a	d 2731–8736 z	d 240–1024 e	d 44–192 z	d 13–32 f	d 8 n
9, 9	a $\mathbf{262144}^{1}$ a	g 9368–12288 g	g 751–1024 g	g 110–256 g	g 26–64 g	g 10–16 g
10, 2	d 19–24 s	4				
10, 3	d 76–112 ζ	d 12 n	4			
10, 4	d 304–508 η	d 24–112 f	d 8 n	4		
10, 5	d 1216–2044 η	d 64–508 f	d 16–24 n	$\mathbf{4}^{1}$		
10, 6	r 4864–11197 ρ	d 228–2044 f	d 32–280 z	d 12–20 n	4	
10, 7	d 16448–53428 z	d 762–9191 f	d 84–1288 z	d 19–48 n	d 9–14 n	4
10, 8	r 65792–116281 m	d 2731–23232 z	d 240–3016 z	d 44–336 z	d 13–36 n	d 8–10 n
10, 9	a 262144 a	d 9368–38496 z	d 751–4096 e	d 110–832 e	d 26–120 n	d 10–24 n
10, 10	a $\mathbf{1048576}^{1}$ a	g 34953–49152 g	g 2412–4096 g	g 313–832 g	g 59–208 g	g 18–54 g

(*continued on next page*)

radius r. Let x be a binary word with s coordinates. Let x_0, x_1 be the results of replacing the last coordinate of x with 0 and 1, respectively. There exist two codewords $c_0, c_1 \in C$ such that $d(x_0, c_0) \le r$ and $d(x_1, c_1) \le r$. Let $d_0, d_1 \in D$ be the results of replacing the last coordinate of c_0 and c_1 with the parity checksum of all coordinates of c_0 and c_1. Now we prove that $d(x, d_0) \le r$ or $d(x, d_1) \le r$. Let x', c_0', c_1' be the binary words with $s-1$ coordinates obtained by deleting the last coordinate of x, c_0 and c_1. If $d(x', c_0') < r$ or $d(x', c_1') < r$, then it is evident that $d(x, d_0) \le r$ or $d(x, d_1) \le r$. Let us suppose that $d(x', c_0') = r$ and $d(x', c_1') = r$. Then c_0' and c_1' must have the same parity and the last coordinates of c_0 and c_1 must be different. Hence the last coordinates of d_0 and d_1 are different. Depending on the value of the last coordinate of x, either $d(x, d_0) = r$ or $d(x, d_1) = r$. $\square$

Now we turn to some special cases. Let C be a $\mathrm{CA}_r(M; s, n, q)$ and let $k_1, k_2, \ldots, k_n$ be arbitrary nonnegative integers. Let $k = \sum_{i=1}^{n} k_i$. Let $C(k_1, k_2, \ldots, k_n)$ be the $M \times k$ array that has k_i identical columns, which are equal to the ith column of C for $i = 1, 2, \ldots, n$.

Table 7 (*continued*)

$\mathrm{CAN}_r(s, n, 5)$					
$n, s \setminus r$	0	1	2	3	4
2, 2	a **25**^{1} a	**5**^{7}			
3, 2	a 25 a	5			
3, 3	a **125**^{1} a	g **13**^{1} g	**5**^{54}		
4, 2	a 25 a	5			
4, 3	a 125 a	t **14**^{1} t	5		
4, 4	a **625**^{1} a	g 46–51 g	g 11 g	**5**^{471}	
5, 2	a 25 a	5			
5, 3	a 125 a	y 15–16 s	5		
5, 4	a 625 a	d 46–70 s	d 11–12 n	5	
5, 5	a **3125**^{1} a	g 160–184 g	g 22–35 g	g 9 g	5
6, 2	a 25 a	5			
6, 3	a 125 a	d 15–18 n	5		
6, 4	a 625 a	d 46–125 f	d 11–13 n	5	
6, 5	a 3125 a	d 160–243 c	d 22–45 n	d 9 n	5
6, 6	a **15625**^{1} a	g 625 g	g 71–125 g	g 16–25 g	**5**^{1}
7, 2	p 29 θ	5			
7, 3	v 135–180 s	d 15–18 n	5		
7, 4	p 626–910 s	d 46–125 f	d 11–13 n	5	
7, 5	p 3126–4375 ϵ	d 160–625 f	d 22–52 n	d 9 n	5
7, 6	a 15625 a	d 625–1220 c	d 71–125 s	d 16–25 d	**5**^{1}
7, 7	a **78125**^{1} a	g 2722–3125 g	g 225–525 g	g 38–100 g	g 12–21 g
8, 2	d 29–33 θ	5			
8, 3	r 145–185 d	d 15–19 n	5		
8, 4	r 675–1245 d	d 46–180 f	d 11–13 n	5	
8, 5	r 3130–5954 ϕ	d 160–910 f	d 22–54 n	d 9 n	5
8, 6	r 15630–27717 τ	d 625–3720 z	d 71–333 z	d 16–25 s	**5**^{1}
8, 7	a 78125 a	d 2722–8005 z	d 225–1025 z	d 38–125 f	d 12–21 n
8, 8	a **390625**^{1} a	g 11945–15625 g	g 821–1625 g	g 109–325 g	g 25–65 g
9, 2	d 29–35 μ	5			
9, 3	d 145–185 d	d 15–19 n	5		
9, 4	r 725–1245 d	d 46–185 f	d 11–13 n	5	
9, 5	r 3375–6996 ϕ	d 160–1245 f	d 22–57 n	d 9 n	5
9, 6	r 15650–35762 ρ	d 625–5954 f	d 71–549 z	d 16–52 n	**5**^{1}
9, 7	r 78150–165625 z	d 2722–22885 z	d 225–2357 z	d 38–183 n	d 12–25 n
9, 8	a 390625 a	d 11945–47645 z	d 821–5725 z	d 109–825 z	d 25–125 h
9, 9	a **1953125**^{1} a	g 53138–78125 g	g 3367–6375 g	g 330–1275 g	g 64–255 g
10, 2	d 29–36 λ	5			
10, 3	d 145–185 ζ	d 15–20 n	5		
10, 4	d 725–1245 η	d 46–185 f	d 11–13 n	5	
10, 5	r 3625–8169 ϕ	d 160–1245 f	d 22–57 n	d 9 n	5
10, 6	r 16875–44368 ρ	d 625–6996 f	d 71–777 z	d 16–53 n	**5**^{1}
10, 7	r 78250–284705 z	d 2722–35762 f	d 225–4553 z	d 38–183 n	d 12–25 n
10, 8	r 390750–944965 z	d 11945–139185 z	d 821–15153 z	d 109–1557 z	d 25–171 n
10, 9	a 1953125 a	d 53138–268705 z	d 3367–23000 e	d 330–3125 e	d 64–543 n
10, 10	a **9765625**^{1} a	g 238993–390625 g	g 13161–23000 g	g 1163–3125 g	g 162–875 g

(*continued on next page*)

Theorem 7.4. *Let*

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then for $k \geq 2$ and k even, $C(k_1, k_2, k_3)$ is a $\mathrm{CA}_{(k-2)/2}(4; k, k, 2)$ if and only if exactly one of k_1, k_2, k_3 is even.

Proof. Consider the covering code whose codewords are the rows of $C(k_1, k_2, k_3)$. Then the assertion is essentially [40, Theorem 9]. □

A straightforward consequence of Theorem 7.4 is the following.

Table 7 (*continued*)

$\mathrm{CAN}_r(s, n, 6)$				
$n, s \setminus r$	0	1	2	3
2, 2	a $\mathbf{36}^{1}$ a	$\mathbf{6}^{11}$		
3, 2	a 36 a	6		
3, 3	a $\mathbf{216}^{1}$ a	g 18 g	6	
4, 2	ω37μ	6		
4, 3	a 216 a	d 18 n	6	
4, 4	a $\mathbf{1296}^{1}$ a	g 72 g	x 15 g	6
5, 2	d 37–39 μ	6		
5, 3	r 222–240 s	d 18–22 n	6	
5, 4	a 1296 a	d 72–108 s	d 15–17 n	6
5, 5	a $\mathbf{7776}^{1}$ a	g 330–414 g	g 36–66 g	g 12 g
6, 2	d 37–41 κ	6		
6, 3	d 222–258 s	d 18–24 n	6	
6, 4	r 1332–1656 s	d 72–218 z	d 15–18 n	6
6, 5	a 7776 a	d 330–721 c	d 36–66 n	d 12–14 n
6, 6	a $\mathbf{46656}^{1}$ a	g 1578–1840 g	g 133–274 g	g 24–41 g
7, 2	p 38–42 κ	6		
7, 3	d 222–293 s	d 18–24 n	6	
7, 4	d 1332–1891 s	d 72–258 f	d 15–18 n	6
7, 5	r 7992–10368 ϵ	d 330–1656 f	d 36–80 n	d 12–14 n
7, 6	a 46656 a	d 1578–4366 c	d 133–446 c	d 24–58 n
7, 7	a $\mathbf{279936}^{1}$ a	g 7777–11040 g	g 528–1296 g	g 70–246 g
8, 2	p 41–42 κ	6		
8, 3	r 228–304 s	d 18–26 n	6	
8, 4	d 1332–2044 s	d 72–293 f	d 15–18 n	6
8, 5	d 7992–16020 σ	d 330–1891 f	d 36–84 n	d 12–14 n
8, 6	r 47952–87818 ϕ	d 1578–10368 f	d 133–846 z	d 24–74 n
8, 7	a 279936 a	d 7777–32870 z	d 528–3526 z	d 70–282 n
8, 8	a $\mathbf{1679616}^{1}$ a	g 41991–62208 g	g 2276–5184 g	g 246–1080 g
9, 2	d 41–46 θ	6		
9, 3	r 246–379 s	d 18–26 n	6	
9, 4	r 1368–2906 σ	d 72–304 f	d 15–18 n	6
9, 5	d 7992–19113 σ	d 330–2044 f	d 36–90 n	d 12–14 n
9, 6	d 47952–115811 ϕ	d 1578–16020 f	d 133–1266 z	d 24–84 f
9, 7	r 287712–590976 z	d 7777–84710 z	d 528–7756 z	d 70–318 n
9, 8	a 1679616 a	d 41991–226558 z	d 2276–22814 z	d 246–2490 z
9, 9	a $\mathbf{10077696}^{1}$ a	g 219096–324864 g	g 10900–29808 g	g 921–4752 g
10, 2	d 41–51 θ	6		
10, 3	d 246–393 s	d 18–28 n	6	
10, 4	r 1476–3287 σ	d 72–379 f	d 15–18 n	6
10, 5	r 8208–22280 σ	d 330–2906 f	d 36–90 n	d 12–14 n
10, 6	d 47952–139638 ϕ	d 1578–19113 f	d 133–1716 z	d 24–90 f
10, 7	d 287712–1071576 z	d 7777–115811 f	d 528–14086 z	d 70–318 n
10, 8	r 1726272–4314156 z	d 41991–590976 f	d 2276–61594 z	d 246–4080 z
10, 9	a 10077696 a	d 219096–1457654 z	d 10900–132480 e	d 921–17202 z
10, 10	a $\mathbf{60466176}^{1}$ a	g 1209324–1866240 g	g 53463–132480 g	g 3815–19347 g
$\mathrm{CAN}_r(s, n, 7)$				
$n, s \setminus r$	0	1	2	3
2, 2	a $\mathbf{49}^{1}$ a	$\mathbf{7}^{15}$		
3, 2	a 49 a	7		
3, 3	a $\mathbf{343}^{1}$ a	g 25 g	7	
4, 2	a 49 a	7		
4, 3	a 343 a	d 25 n	7	
4, 4	a $\mathbf{2401}^{1}$ a	g 115–123 g	g 17–19 g	7
5, 2	a 49 a	7		
5, 3	a 343 a	d 25–27 n	7	
5, 4	a 2401 a	d 115–232 c	d 17–22 n	7
5, 5	a $\mathbf{16807}^{1}$ a	g 606–769 g	g 55–97 g	x 15–17 g

(*continued on next page*)

Table 7 (*continued*)

6, 2	a 49 a	7		
6, 3	a 343 a	d 25–31 n	7	
6, 4	a 2401 a	d 115–343 f	d 17–24 n	7
6, 5	a 16807 a	d 606–1815 c	d 55–97 n	d 15–19 n
6, 6	a **117649**[1] a	g 3412–4435 g	g 233–343 g	g 36–77 g
7, 2	a 49 a	7		
7, 3	a 343 a	d 25–33 n	7	
7, 4	a 2401 a	d 115–343 f	d 17–24 n	7
7, 5	a 16807 a	d 606–2401 f	d 55–128 n	d 15–19 n
7, 6	a 117649 a	d 3412–12839 c	d 233–925 z	d 36–97 f
7, 7	a **823543**[1] a	g 19818–31045 g	g 1035–2401 g	g 127–343 g
8, 2	a 49 a	7		
8, 3	a 343 a	d 25–35 n	7	
8, 4	a 2401 a	d 115–343 f	d 17–25 n	7
8, 5	a 16807 a	d 606–2401 f	d 55–130 n	d 15–19 n
8, 6	a 117649 a	d 3412–16807 f	d 233–1693 z	d 36–120 n
8, 7	a 823543 a	d 19818–108079 z	d 1035–7951 z	d 127–649 n
8, 8	a **5764801**[1] a	g 117649 g	g 5457–15129 g	g 457–2337 g
9, 2	p 55–59 s	7		
9, 3	v 354–472 s	d 25–36 n	7	
9, 4	d 2401–4095 q	d 115–343 f	d 17–25 n	7
9, 5	d 16807–30870 u	d 606–2401 f	d 55–157 n	d 15–19 n
9, 6	d 117649–216090 u	d 3412–16807 f	d 233–2401 f	d 36–130 f
9, 7	p 823544–1512630 u	d 19818–117649 f	d 1035–16807 f	d 127–667 n
9, 8	a 5764801 a	d 117649–766123 z	d 5457–62835 z	d 457–6231 z
9, 9	a **40353607**[1] a	g 733726–823543 g	g 29889–94587 g	g 2077–8575 g
10, 2	d 55–61 θ	7		
10, 3	r 385–479 s	d 25–38 n	7	
10, 4	r 2478–4795 η	d 115–472 f	d 17–26 n	7
10, 5	d 16807–45276 z	d 606–4095 f	d 55–157 n	d 15–19 n
10, 6	d 117649–316932 z	d 3412–30870 f	d 233–2401 f	d 36–157 f
10, 7	d 823544–2218524 z	d 19818–216090 f	d 1035–16807 f	d 127–667 n
10, 8	r 5764808–10706059 z	d 117649–1472017 z	d 5457–117649 f	d 457–10233 z
10, 9	a 40353607 a	d 733726–5420281 z	d 29889–420175 e	d 2077–42189 e
10, 10	a **282475249**[1] a	g 4630843–5764801 g	g 168042–420175 g	g 10577–42189 g

(*continued on next page*)

Corollary 7.5. *$C(k_1, k_2, k_3)$ is a $\mathrm{CA}_{(k-3)/2}(4; k-1, k, 2)$ if all of k_1, k_2, k_3 are odd, and consequently*

$$\mathrm{CAN}_{(k-3)/2}(k-1, k, 2) = 4 \quad \text{for } k \geq 3 \text{ odd}. \tag{13}$$

Theorems 7.6, 7.8 and 7.10, and their corollaries, are analogous to Theorem 7.4 and its corollary, for binary 2-surjective codes with 7, 12, and 16 codewords.

Theorem 7.6. *Let*

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then for odd $k \geq 5$, $C(k_1, k_2, k_3, k_4, k_5, k_6)$ is a $\mathrm{CA}_{(k-3)/2}(7; k, k, 2)$ if and only if exactly one of $k_1, k_2, k_3, k_4, k_5, k_6$ is even.

Proof. The assertion is equivalent to [28, Theorem 3.2]. □

Corollary 7.7. *$C(k_1, k_2, k_3, k_4, k_5, k_6)$ is a $\mathrm{CA}_{(k-4)/2}(7; k-1, k, 2)$ if $k_1, k_2, k_3, k_4, k_5, k_6$ are all odd, and consequently*

$$\mathrm{CAN}_{(k-4)/2}(k-1, k, 2) = 7 \text{ for } k \geq 6 \text{ even}. \tag{14}$$

Table 7 (*continued*)

$\mathrm{CAN}_r(s, n, 8)$				
$n, s \setminus r$	0	1	2	3
2, 2	a **64**[1] a	**8**[22]		
3, 2	a 64 a	8		
3, 3	a **512**[1] a	g 32 g	8	
4, 2	a 64 a	8		
4, 3	a 512 a	d 32 n	8	
4, 4	a **4096**[1] a	g 171–192 g	g 22–23 g	8
5, 2	a 64 a	8		
5, 3	a 512 a	d 32 n	8	
5, 4	a 4096 a	d 171–416 z	d 22–27 n	8
5, 5	a **32768**[1] a	g 1024 g	g 83–128 g	g 17–22 g
6, 2	a 64 a	8		
6, 3	a 512 a	d 32–37 n	8	
6, 4	a 4096 a	d 171–512 f	d 22–30 n	8
6, 5	a 32768 a	d 1024–3936 z	d 83–128 n	d 17–24 n
6, 6	a **262144**[1] a	g 6626–8192 g	g 382–512 g	g 52–107 g
7, 2	a 64 a	8		
7, 3	a 512 a	d 32–41 n	8	
7, 4	a 4096 a	d 171–512 f	d 22–30 n	8
7, 5	a 32768 a	d 1024–4096 f	d 83–176 n	d 17–24 n
7, 6	a 262144 a	d 6626–32690 c	d 382–1408 z	d 52–128 f
7, 7	a **2097152**[1] a	g 44237–63488 g	g 1984–4096 g	g 196–512 g
8, 2	a 64 a	8		
8, 3	a 512 a	d 32–44 n	8	
8, 4	a 4096 a	d 171–512 f	d 22–32 n	8
8, 5	a 32768 a	d 1024–4096 f	d 83–176 n	d 17–24 n
8, 6	a 262144 a	d 6626–32768 f	d 382–2640 z	d 52–176 f
8, 7	a 2097152 a	d 44237–262144 f	d 1984–13952 z	d 196–1016 n
8, 8	a **16777216**[1] a	g 302036–342272 g	g 11766–29920 g	g 829–4096 g
9, 2	a 64 a	8		
9, 3	a 512 a	d 32–46 n	8	
9, 4	a 4096 a	d 171–512 f	d 22–32 n	8
9, 5	a 32768 a	d 1024–4096 f	d 83–224 n	d 17–24 n
9, 6	a 262144 a	d 6626–32768 f	d 382–3872 z	d 52–176 f
9, 7	a 2097152 a	d 44237–262144 f	d 1984–32432 z	d 196–1016 n
9, 8	a 16777216 a	d 302036–2097152 e	d 11766–127584 z	d 829–11208 z
9, 9	a **134217728**[1] a	g 2097152 g	g 75783–174080 g	g 4523–16384 g
10, 2	p 67–72 ψ	8		
10, 3	a 512 a	d 32–48 n	8	
10, 4	p 4097–6560 q	d 171–512 f	d 22–34 n	8
10, 5	d 32768–59048 q	d 1024–4096 f	d 83–224 n	d 17–24 n
10, 6	d 262144–487424 u	d 6626–32768 f	d 382–4096 f	d 52–224 f
10, 7	d 2097152–3899392 u	d 44237–262144 f	d 1984–32768 f	d 196–1016 n
10, 8	p 16777217–31195136 u	d 302036–2097152 f	d 11766–262144 f	d 829–18320 z
10, 9	a 134217728 a	d 2097152–16777216 e	d 75783–1048576 e	d 4523–94840 z
10, 10	a **1073741824**[1] a	g 15339169–16777216 g	g 478586–1048576 g	g 25767–98304 g

Theorem 7.8. *Let*

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that exactly one of $\{k_1, k_2, k_3, k_4, k_5, k_6, k_7\}$ *is even and hence that* $k = \sum_{i=1}^{7} k_i \geq 6$. *Then* $C(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$ *is a* $\mathrm{CA}_{(k-4)/2}(12; k, k, 2)$.

Proof. Consider the code whose codewords c_i are the rows of $C(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$. Let $x = |x_1|x_2|x_3|x_4|x_5|x_6|x_7|$ be an arbitrary word in Z_2^k partitioned according to the integers $k_1, k_2, k_3, k_4, k_5, k_6, k_7$, and let w_i denote the weight of x_i, i.e., the number of 1-s in it. Then the Hamming distances of x from the codewords are

$$
\begin{aligned}
&d(x, c_1) = w_1 + w_2 + w_3 + w_4 + w_5 + (k_6 - w_6) + (k_7 - w_7),\\
&d(x, c_2) = w_1 + w_2 + w_3 + w_4 + (k_5 - w_5) + w_6 + (k_7 - w_7),\\
&d(x, c_3) = w_1 + w_2 + w_3 + (k_4 - w_4) + w_5 + w_6 + (k_7 - w_7),\\
&d(x, c_4) = w_1 + w_2 + (k_3 - w_3) + (k_4 - w_4) + (k_5 - w_5) + (k_6 - w_6) + w_7,\\
&d(x, c_5) = w_1 + (k_2 - w_2) + w_3 + (k_4 - w_4) + (k_5 - w_5) + (k_6 - w_6) + w_7,\\
&d(x, c_6) = (k_1 - w_1) + w_2 + w_3 + (k_4 - w_4) + (k_5 - w_5) + (k_6 - w_6) + w_7,\\
&d(x, c_7) = (k_1 - w_1) + (k_2 - w_2) + (k_3 - w_3) + w_4 + (k_5 - w_5) + (k_6 - w_6) + (k_7 - w_7),\\
&d(x, c_8) = (k_1 - w_1) + (k_2 - w_2) + (k_3 - w_3) + (k_4 - w_4) + w_5 + (k_6 - w_6) + (k_7 - w_7),\\
&d(x, c_9) = (k_1 - w_1) + (k_2 - w_2) + (k_3 - w_3) + (k_4 - w_4) + (k_5 - w_5) + w_6 + (k_7 - w_7),\\
&d(x, c_{10}) = w_1 + (k_2 - w_2) + (k_3 - w_3) + w_4 + w_5 + w_6 + w_7,\\
&d(x, c_{11}) = (k_1 - w_1) + w_2 + (k_3 - w_3) + w_4 + w_5 + w_6 + w_7,\\
&d(x, c_{12}) = (k_1 - w_1) + (k_2 - w_2) + w_3 + w_4 + w_5 + w_6 + w_7,
\end{aligned}
$$

and consequently

$$
d(x, C) \leq \frac{\sum_{i=1}^{12} d(x, c_i)}{12} = \frac{6\sum_{i=1}^{7} k_i}{12} = k/2.
$$

First, we prove that $d(x, c_j) \leq (k-4)/2$ for at least one codeword. Let us assume to the contrary that $d(x, c_j) \geq (k-3)/2$ for each codeword. Then because k is even, $d(x, c_j) \geq (k-2)/2$ for each codeword. Introducing the variables $y_i = 2w_i - k_i$, we get the following system of inequalities.

$$
\begin{aligned}
&y_1 + y_2 + y_3 + y_4 + y_5 - y_6 - y_7 + 2 \geq 0\\
&y_1 + y_2 + y_3 + y_4 - y_5 + y_6 - y_7 + 2 \geq 0\\
&y_1 + y_2 + y_3 - y_4 + y_5 + y_6 - y_7 + 2 \geq 0\\
&y_1 + y_2 - y_3 - y_4 - y_5 - y_6 + y_7 + 2 \geq 0\\
&y_1 - y_2 + y_3 - y_4 - y_5 - y_6 + y_7 + 2 \geq 0\\
&-y_1 + y_2 + y_3 - y_4 - y_5 - y_6 + y_7 + 2 \geq 0\\
&-y_1 - y_2 - y_3 + y_4 - y_5 - y_6 - y_7 + 2 \geq 0\\
&-y_1 - y_2 - y_3 - y_4 + y_5 - y_6 - y_7 + 2 \geq 0\\
&-y_1 - y_2 - y_3 - y_4 - y_5 + y_6 - y_7 + 2 \geq 0\\
&y_1 - y_2 - y_3 + y_4 + y_5 + y_6 + y_7 + 2 \geq 0\\
&-y_1 + y_2 - y_3 + y_4 + y_5 + y_6 + y_7 + 2 \geq 0\\
&-y_1 - y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + 2 \geq 0.
\end{aligned}
$$

As y_i and k_i have the same parity, exactly one among the y_i is even. The value of each y_i may be either nonnegative or negative. Now we show that the set defined by the given system of inequalities is bounded. The boundedness of y_7 follows from

$$
\frac{k-2}{2} \leq d(x, C) \leq \frac{d(x, c_1) + d(x, c_9)}{2} = \frac{k + k_7 - 2w_7}{2} = \frac{k - y_7}{2}
$$

and

$$
\frac{k-2}{2} \leq d(x, C) \leq \frac{d(x, c_4) + d(x, c_{12})}{2} = \frac{k + 2w_7 - k_7}{2} = \frac{k + y_7}{2}.
$$

Then, the existence of a lower bound on y_1 follows from

$$
\begin{aligned}
\frac{k-2}{2} &\leq d(x, C)\\
&\leq \frac{3d(x, c_1) + d(x, c_2) + d(x, c_3) + 2d(x, c_4) + 2d(x, c_5) + 2d(x, c_9) + 3d(x, c_{10})}{14}\\
&= \frac{7k + 10w_1 - 5k_1}{14} = \frac{7k + 5y_1}{14},
\end{aligned}
$$

which yields $y_1 \geq -2$. The upper bound $y_1 \leq 2$ and the bounds $-2 \leq y_i \leq 2$ for the other variables can be proved similarly.

Knowing that $|y_i| \leq 2$ for all variables y_i, a simple computer program proves that the system of inequalities under consideration has no such solutions in integers where exactly one of the variables has an even value. By this, it is proved that $d(x, C) \leq (n-4)/2$.

In order to show that $d(x, C) \geq (n-4)/2$ is possible, and hence the covering radius cannot be smaller than $(n-4)/2$, choose a word x so that $w_i = \lfloor k_i/2 \rfloor$ for all $i = 1, 2, \ldots, 7$. Then $d(x, C) = (n-4)/2$. □

Corollary 7.9. *$C(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$ is a $\mathrm{CA}_{(k-5)/2}(12; k-1, k, 2)$ if all of $k_1, k_2, k_3, k_4, k_5, k_6, k_7$ are odd, and consequently*

$$\mathrm{CAN}_{(k-5)/2}(k-1, k, 2) \leq 12 \quad \textit{for } k \geq 7 \textit{ odd}. \tag{15}$$

An analogous assertion with sixteen-row arrays can be formulated by using an array C made from the binary extended Hamming code with seven coordinates and parity check bits in the eighth coordinate.

Theorem 7.10. *Let*

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0\\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0\\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1\\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1\\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

Now assume that exactly one of $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ is even, and hence $k = \sum_{i=1}^{8} k_i \geq 7$. Then $C(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ is a $\mathrm{CA}_{(k-5)/2}(16; k, k, 2)$.

Proof. To prove that $d(x, c_j) \leq (k-5)/2$ for at least one codeword, where $x = |x_1|x_2|x_3|x_4|x_5|x_6|x_7|x_8|$ is an arbitrary word, consider the variables $y_i = 2w_i - k_i$. Exactly as in the proof of Theorem 7.8, using the Hamming distances of x from the rows c_j of $C(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$, the analogous argumentation leads to the claim that the system of inequalities

$$\begin{aligned}
&y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + 3 \geq 0\\
&y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 - y_8 + 3 \geq 0\\
&y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 - y_8 + 3 \geq 0\\
&y_1 + y_2 - y_3 - y_4 - y_5 - y_6 + y_7 + y_8 + 3 \geq 0\\
&y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 - y_8 + 3 \geq 0\\
&y_1 - y_2 + y_3 - y_4 - y_5 + y_6 - y_7 + y_8 + 3 \geq 0\\
&y_1 - y_2 - y_3 + y_4 + y_5 - y_6 - y_7 + y_8 + 3 \geq 0\\
&y_1 - y_2 - y_3 - y_4 + y_5 + y_6 + y_7 - y_8 + 3 \geq 0\\
&-y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7 + y_8 + 3 \geq 0\\
&-y_1 + y_2 + y_3 - y_4 - y_5 + y_6 + y_7 - y_8 + 3 \geq 0\\
&-y_1 + y_2 - y_3 + y_4 + y_5 - y_6 + y_7 - y_8 + 3 \geq 0\\
&-y_1 + y_2 - y_3 - y_4 + y_5 + y_6 - y_7 + y_8 + 3 \geq 0\\
&-y_1 - y_2 + y_3 + y_4 + y_5 + y_6 - y_7 - y_8 + 3 \geq 0\\
&-y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7 + y_8 + 3 \geq 0\\
&-y_1 - y_2 - y_3 + y_4 - y_5 + y_6 + y_7 + y_8 + 3 \geq 0\\
&-y_1 - y_2 - y_3 - y_4 - y_5 - y_6 - y_7 - y_8 + 3 \geq 0
\end{aligned}$$

has no integer solution such that exactly one of the variables y_i is even.

To prove this, we start again with the examination of the boundedness of the set of all integer solutions. Summing the first eight lines, and then distinctly the remaining eight lines, results in the proof that $|y_1| \leq 3$.

The boundedness of the other variables can be proved similarly. Now, the proof of this part can be completed by a computer search on the set determined by $|y_i| \leq 3$ for all y_i having the suitable parities.

To prove the possibility of $d(x, C) \geq (k-5)/2$, consider a word x so that $w_i = \lceil k_i/2 \rceil$ for a unique i belonging to an odd k_i, and $w_i = \lfloor k_i/2 \rfloor$ for all other i. $\square$

Corollary 7.11. *$C(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ is a $\mathrm{CA}_{(k-6)/2}(16; k-1, k, 2)$ if all of $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ are odd, and consequently*

$$\mathrm{CAN}_{(k-6)/2}(k-1, k, 2) \leq 16 \quad \text{for } k \geq 8 \text{ even}. \tag{16}$$

8. Tables: Lower and upper bounds, and classifications

Extensive tables of upper bounds when $r = 0$ appear at [14]. We provide more detailed tables for a smaller range of parameters, in order to collect together best known lower and upper bounds with specific references for each, along with the classification results. At the same time, we tabulate information concerning radius-covering arrays. The tables here treat $2 \leq q \leq 8$ and $2 \leq n \leq 14$ for the binary case, and $2 \leq n \leq 10$ for the nonbinary cases. The range for r depends on the values of the other parameters. For an entry of these tables, bold typesetting and a superscript indicates that a classification result is available, and the value in the superscript gives the number of inequivalent covering arrays.

In addition to the basic upper bounds developed in Section 3, we employ a set of basic lower bounds, which apply for all $q \geq 2$.

$$\begin{aligned}
&\mathrm{CAN}(2, q+1, q) \geq q^2+2 \quad \text{if } \mathrm{CAN}(2, q+1, q) \neq q^2 \text{ [19]}\\
&\mathrm{CAN}(2, q+2, q) \geq q^2+q-1 \quad \text{if } 2 \leq q \leq 7 \text{ [45]}\\
&\mathrm{CAN}(2, q+2, q) \geq q^2+3 \quad \text{if } q \geq 8 \text{ [46]}\\
&\mathrm{CAN}(s, q+s-1, q) \geq q^s+1 \quad \text{if } q \equiv 1 \pmod 2,\ 3 \leq s < q \text{ [8]}\\
&\mathrm{CAN}(s, q+s, q) \geq q^s+1 \quad \text{if } q \equiv 0 \pmod 2,\ 3 \leq s < q \text{ [8]}\\
&\mathrm{CAN}(s, s+2, q) \geq q^s+1 \quad \text{if } s \geq q \text{ [8]}\\
&\mathrm{CAN}(3, q+1, q) \geq q^3+1 \quad \text{if } q \equiv 2 \pmod 4,\ q \geq 6 \text{ [31]}\\
&\mathrm{CAN}(s, q+s-2, q) \geq q^s+1 \quad \text{if } q \equiv 0 \pmod 2,\ q \not\equiv 0 \pmod{36},\ 4 \leq s < q \text{ [31]}\\
&\mathrm{CAN}(4, 7, 5) \geq 5^4+1 \text{ [31]}.
\end{aligned} \tag{17}$$

The lower bounds implied by (17) can be improved upon in certain cases by counting arguments (Table 7).

Theorem 8.1.

$$\mathrm{CAN}(3, q+2, q) \geq q^3 + \begin{cases} 6 & \text{if } q = 3 \\ 10 & \text{if } q = 5 \\ 11 & \text{if } q \in \{7, 9\} \\ 12 & \text{if } q \equiv 1 \pmod 2,\ q \geq 11. \end{cases} \tag{18}$$

Proof. Suppose that there is a $\mathrm{CA}(q^3+a; 3, q+2, q)$ with $a < 2q$. Then some column contains a symbol σ so that the number of rows with σ in the chosen column is either q^2 or q^2+1. Delete this column, and call the rows that contained σ the *plane rows*. The choice of name is explained as follows. Suppose that the $q+1$ remaining columns are indexed by R, and form a set of q^2+q points, $R \times \{0, \ldots, q-1\}$. Each of the plane rows selects a $(q+1)$-set of these points, which we call a *line*. Adjoin a point ∞ and the lines $\{\infty\} \cup (\{i\} \times \{0, \ldots, q-1\})$ for every $i \in R$. Then we have formed q^2+q+1 or q^2+q+2 lines on q^2+q+1 points in which every pair of points lies on at least one line; when the number of lines is q^2+q+1, every pair lies on exactly one line and this is a *projective plane* of order q. When the number is q^2+q+2, the design covers all pairs, some more than once. In this case, Füredi [19] shows that q^2+q+1 of the lines form a projective plane, so we employ only those lines as plane rows henceforth. Now consider a row of the covering array that does not arise from a plane row, and let its values in the remaining columns be $(\sigma_i : i \in R)$. Let $S = \{\infty\} \cup \{(i, \sigma_i) : i \in R\}$ be a set of $q+2$ points. Because q is odd, there is a line of the plane containing at least three points in S (otherwise, S is a hyperoval) [5]. By construction, S contains at most two from each line containing ∞, and hence S must intersect one of the plane rows in three or more points. The number of triples that a non-plane row covers that are not already covered in one or more plane rows cannot exceed $\binom{q+1}{3} - 1$. The number of triples not covered by plane rows is $\binom{q+1}{3} q^2(q-1)$. Hence

$$\mathrm{CAN}(3, q+2, q) \geq q^2 + \left\lceil \frac{\binom{q+1}{3} q^2(q-1)}{\binom{q+1}{3} - 1} \right\rceil = q^3 + 6 - \left\lfloor \frac{6q^2-6q-36}{q^3-q-6} \right\rfloor.$$

Then $\mathrm{CAN}(3, q+2, q) \geq q^3 + \min(2q, 6)$ follows directly.

When $q > 3$ we can obtain a better bound. Suppose that the set S has exactly one line meeting it in three points $\{x, y, z\}$, and all others meeting it in at most two. Then removing any of x, y, or z yields an *oval*, a set of $q + 1$ points with every line meeting it in at most two points. By [17, Section 3.2.25(a)], provided that $S \setminus \{x, y\}$ has more than $(q + 3)/2$ points, it completes to a unique oval. But then $S \setminus \{x\} = S \setminus \{y\}$, a contradiction. Thus when $q > (q+3)/2$, any set of $q+2$ points must either have a line intersect it in four or more points, or at least two lines each intersect it in three points. Then for $q \geq 5$, the number of triples that a non-plane row covers that are not already covered in one or more plane rows cannot exceed $\binom{q+1}{3} - 2$. Hence

$$\mathrm{CAN}(3, q+2, q) \geq q^2 + \left\lceil \frac{\binom{q+1}{3} q^2(q-1)}{\binom{q+1}{3} - 2} \right\rceil = q^3 + 12 - \left\lfloor \frac{12q^2 - 12q - 144}{q^3 - q - 12} \right\rfloor . \quad \square$$

The following two inequalities that proved to be useful for setting lower bounds on the size of covering codes were published in [2,26], respectively.

$$K_q(n_1 + n_2, r_1 + r_2 + 1) \geq \min\{K_q(n_1, r_1), K_q(n_2, r_2)\}, \tag{19}$$

$$K_q(n + s, R + r + 1) \geq \min\{\mathrm{CAN}_r(s, n + s, q), K_q(n, R)\}. \tag{20}$$

The analogous extension of these inequalities, proved in [41], can be used for radius-covering arrays:

Theorem 8.2 (*[41, Theorem 5]*).

$$\mathrm{CAN}_{r_1+r_2+1}(s_1 + s_2, n_1 + n_2, q) \geq \min\{\mathrm{CAN}_{r_1}(s_1, n_1, q), \mathrm{CAN}_{r_2}(s_2, n_2, q)\}, \tag{21}$$

$$\mathrm{CAN}_{R+r+1}(n + s, n + k + s, q) \geq \min\{\mathrm{CAN}_r(s, n + k + s, q), \mathrm{CAN}_R(n, n + k, q)\}. \tag{22}$$

Acknowledgements

Thanks are due to Tim Penttila for useful comments on finite geometry. Research of the first author is supported by DOD grants N00014-08-1-1069 and N00014-08-1-1070. Research of the second author is partly supported by OTKA grants K 60480 and K 77420.

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